

Heat flow in anharmonic crystals with internal and external stochastic baths: A convergent polymer expansion for a model with discrete time and long range interparticle interaction

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We investigate a chain of oscillators with anharmonic on-site potentials, with long range interparticle interactions, and coupled both to external and internal stochastic thermal reservoirs of Ornstein-Uhlenbeck type. We develop an integral representation, a la Feynman-Kac, for the correlations and the heat current. We assume the approximation of discrete times in the integral formalism (together with a simplification in a subdominant part of the harmonic interaction) and develop a suitable polymer expansion for the model. In the regime of strong anharmonicity, strong harmonic pinning, and for the interparticle interaction with integrable polynomial decay, we prove the convergence of the polymer expansion uniformly in volume (number of sites and time). We also show that the two-point correlation decays in space such as the interparticle interaction. The existence of a convergent polymer expansion is of practical interest: it establishes a rigorous support for a perturbative analysis of the heat flow problem and for the computation of the thermal conductivity in related anharmonic crystals, including those with inhomogeneous potentials and long range interparticle interactions. To show the usefulness and trustworthiness of our approach, we compute the thermal conductivity of a specific anharmonic chain, and make a comparison with related numerical results presented in the literature.

I. INTRODUCTION

The derivation of the macroscopic laws of heat transport from the underlying microscopic Hamiltonian models is still a challenge in non-equilibrium statistical mechanics [1]. Since the pioneering work of Debye [2] and Peierls [3], the microscopic models recurrently used to describe heat flow in solids and crystals are mainly given by lattices of anharmonic oscillators, which lead to problems of considerable mathematical difficulty. Consequently, most of the works on the subject are carried out by means of computer simulations [4, 5]. There are, however, some few mathematical results considering different aspects of the anharmonic heat flow problem: for example, there are rigorous works considering the existence of non-equilibrium stationary state [6–8]; the rate of divergence of the thermal conductivity with the system size in Fermi-Pasta-Ulam type models [9]; the on-set of Fourier's law in lattices with anharmonic on-site potentials [10–12]; the finiteness or infiniteness of the thermal conductivity given by a Green-Kubo formula [8, 13].

In systems with normal heat transport, the establishment of bounds showing the finiteness of the thermal conductivity is already an interesting and intricate problem. However, the derivation of more precise expressions, which seems to be an exceedingly difficult task, is highly desirable, both for fundamental reasons as well as in order to provide useful information about properties of the heat conduction with experimental applications, such as the possibility of thermal rectification, the existence of negative differential thermal resistance, etc. In particular, concerning possible applications, it is worth to recall the progress of Phononics [5], the counterpart of electronics devoted to the manipulation and control of the heat current. A considerable effort has been dedicated to the development of Phononic devices, systems idealized to work as electronic analogs, such as thermal diodes, thermal transistors, etc. The basic phenomenon behind the operation of these devices is the thermal rectification, which means asymmetric heat flow, and its understanding involves the investigation of inhomogeneous, asymmetric materials. There is an intense research in this subject, but, again, most of the results are obtained by means of computer simulations or numerical techniques, and so, a profitable analytical approach is opportune.

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In the present work, we aim to develop an approach that allows the detailed study of the heat flow and that can be used in the computation of an expression for the thermal conductivity in reasonable approximations of recurrent lattice models of anharmonic oscillators. Namely, we consider the chain of oscillators with anharmonic on-site potentials, nonlocal interparticle interactions and with stochastic baths coupled to each site: a model in which the time evolution is given by a combination of deterministic and stochastic dynamics. We develop an integral representation (a la Feynman-Kac) for the correlation functions, which are related to the heat flow and to the thermal conductivity. To make treatable the analysis, we introduce a discrete time regularization, i.e., an ultraviolet cutoff. To be free of huge subdominant terms and avoid unimportant technical difficulties, we also consider a simplified expression for the subdominant harmonic interaction. For this discrete time and simplified version, we present a suitable polymer (cluster) expansion, and prove its convergence uniformly in volume (arbitrary number of sites and arbitrary times). We consider a system with interparticle interactions beyond nearest neighbor sites: precisely, our approach is also valid for interparticle potentials with integrable polynomial decay. For these systems with long range interactions, we show that the two-point correlation function, which is directly related to the heat current and the thermal conductivity, decays in space such as the interparticle interaction.

We must emphasize that the existence of an integral representation for the correlations and a convergent cluster expansion for such representation, the main technical achievement of the present work, is of practical importance: it allows an accurate perturbative investigation of the heat flow problem in these systems with anharmonic oscillators. Precisely, the convergence of the expansion proves that the terms with small polymers and with small sizes, which correspond to the terms of lower order in a “naive” perturbative expansion for the potential interaction, contain already the main information about the model. That is, a convergent polymer expansion appears as a support for the validity of some theoretical results previously obtained by means of non rigorous perturbative computations [14–18], and may provide a precise tool for further investigation of the heat flow problem, even in more intricate systems such as inhomogeneous, graded chains or models with long range interparticle interactions. In specific, for these much more complicate models with long range interactions, we recall that present technology permits the fabrication of such systems. For example, nowadays nanomagnets of Permalloy are lithographically manipulated to present interesting properties [19]: these materials are intensively investigated and their interparticle interactions present polynomial decay (such as that considered in the present work).

We stress that the model to be investigated here, given by a chain of anharmonic oscillators (with inner noises, representing extra effective interactions missing in the Hamiltonian), is a natural model for the investigation of the heat conduction in the nonequilibrium steady state of electric insulating solids submitted to different temperatures. As a motivation to understand the importance and usefulness of our approach and results for the (necessary and difficult) analytical investigation of such anharmonic systems, and also as a justification for the discrete approximation to be used here, we recall some recent and important related works carried out within considerable approximations. A very elaborate work involving lattices of anharmonic oscillators is that due to Bricmont and Kupiainen [10, 11]. In these articles, restricted to systems with space dimension $d \geq 3$, the authors take a system with baths at the boundaries only and derive the Fourier’s law for the case of a quartic on-site potential. As well known, Fourier’s law is the phenomenological basic law for the heat transport which states that the heat flow is proportional to temperature gradient. They show that the correlation functions of the system satisfy an infinite set of linear equations (Hopf equations), and, to carry out the investigation, they have to assume a closure approximation to these equations (according to the authors: “this is an uncontrolled approximation that we do not know how to justify rigorously” [11]). To emphasize, again, the difficulty of the analytical study, we quote some other comments: “despite its fundamental nature, a derivation of Fourier’s law from first principles lies well beyond what can be mathematically proven” [10]; “a first principle derivation of the law is missing and, many would say, is not even on the horizon” [11]. Another elaborate and interesting related work is that due to Olla and collaborators [13]. There, to investigate the relation between normal heat transport and space dimension in systems with momentum conservation, the authors consider a hypothetical mathematical model given by harmonic oscillators but perturbed by a nonlinear stochastic dynamics conserving momentum and energy. And the authors say: “a rigorous treatment of a non-linear system, even the proof of the conductivity coefficient, is out of reach of current mathematical techniques” [13].

Given such scenario, in this present paper, in order to perform rigorous analytical investigations in this basic and recurrent model given by anharmonic lattice of oscillators, we also have to assume an approximation: namely, after establishing a rigorous integral representation for the correlations (related to the heat flow), we simplify the dynamics by assuming the evolution given by discrete times. Within such time discretization, we are able to rewrite the integral representation, which originally involves a Gaussian measure, in terms of a new intricate measure with anharmonic terms. We stress that it is a central point: by starting with this new “correct” measure (which really involves the anharmonic interaction part), we are able to develop a convergent polymer expansion, or, in other words, we can perform a rigorous (convergent) perturbative analysis in the problem, which seems impossible with the original Gaussian measure. Of course, the exceeding difficulty of the original problem remains: we do not know how to control the limit of time discretization going to zero, i.e., we cannot treat the original problem with continuous time. This

complication in recovering the continuous limit from a discrete version is very common in physical problems with intricate interactions: recall, for example, the very arduous study of the ultraviolet limit of models in Quantum Field Theory (QFT) with a cutoff [20]. Here, we control the infrared limit of the model: i.e., we develop the polymer expansion and obtain uniform bounds, which are valid even if the space and time volume goes to infinite. Finally, we must emphasize that our formalism with discrete times gives a very accurate result in a comparison with well known numerical works. Precisely, when applied to the computation of the thermal conductivity dependence on temperature for the anharmonic chain with quartic on-site potential. That is, we believe to have a good approach for this difficult problem of anharmonic oscillators. Moreover, as already said, our formalism is also extended to intricate systems with long range interactions, systems with important physical properties.

We organize this paper as follows. In section II, we introduce the model and develop an integral representation for the correlations, that are directly related to the heat flow. In this integral formalism, we still introduce the time discretization and a simplification for part of the harmonic interaction (simplification that is discussed in appendix A). In section III, we introduce a polymer expansion for the model. We prove the convergence of such polymer expansion in section IV. In section V, we show the convergence for the modified polymer expansion associated to the two-point correlation. A concrete example, which makes clear the direct application of our results in the detailed computation of the thermal conductivity, is presented in section VI. Section VII is devoted for final remarks.

II. THE MODEL AND THE INTEGRAL REPRESENTATION

We describe our anharmonic crystal model. We consider a lattice system with unbounded variables, coupled to both external and internal heat baths of Ornstein-Uhlenbeck type. More precisely, we take a system of N oscillators with Hamiltonian

$$H(q, p) = \sum_{j=1}^N \left[\frac{1}{2} \left(p_j^2 + M_j q_j^2 + \sum_{l \neq j} q_l J_{lj} q_j \right) + \lambda P(q_j) \right], \quad (1)$$

where q and p are vectors in \mathbb{R}^N ; $\lambda, M_j > 0$; $J_{jl} = J_{lj} = f(|l - j|)$, f with some integrable decay (details ahead); P is the anharmonic on-site potential, which we take as the polynomial $P(q_j) = q_j^4/4$. For simplicity, we take here the particle masses as 1 (but our method follows also for general cases, including inhomogeneous distributions for the particle masses), and we assume the space dimension $d = 1$. We take the dynamics given by the stochastic differential equations

$$dq_j = p_j dt, \quad dp_j = -\frac{\partial H}{\partial q_j} dt - \zeta_j p_j dt + \gamma_j^{1/2} dB_j, \quad (2)$$

where B_j are independent Brownian motions, with zero average and diffusion equal to 1

$$\langle B_j(t) \rangle = 0, \quad \langle B_j(t) B_\ell(s) \rangle = \delta_{j,\ell} \min(t, s), \quad (3)$$

ζ_j is the constant coupling between site j and its reservoir; $\gamma_j = 2\zeta_j T_j$, where T_j is the temperature of the j -th bath.

To study the heat flow, we first define the energy of a single oscillator

$$H_j(q_j, p_j) = \frac{p_j^2}{2} + \frac{1}{2} \sum_{l \neq j} V(q_l - q_j) + V_2(q_j), \quad (4)$$

where $H(q, p) = \sum_j H_j(q_j, p_j)$. The expression for V comes after writing the interparticle potential in the Hamiltonian above as $\frac{1}{2} \sum_{\ell \neq j} V(q_j - q_\ell) = \frac{1}{2} \sum_{\ell \neq j} J_{j\ell} (q_j - q_\ell)^2$ (with adjustments in M_j , the pinning constant, coefficient of q_j^2); V_2 describes the on-site potentials above and involves the terms $\lambda P(q_j) + M_j q_j^2$ plus some terms with q_j^2 , which appear as we write $q_j J_{j,k} q_k$ as $J_{j,k} q_j q_k / 2$ (as said, these terms may be treated as an adjustment in M_j). Thus, we have

$$\left\langle \frac{dH_j}{dt}(t) \right\rangle = \mathcal{R}_j(t) + \langle \mathfrak{F}_{\rightarrow j} - \mathfrak{F}_{j \rightarrow} \rangle, \quad (5)$$

$$\mathfrak{F}_{j \rightarrow} = \sum_{\ell > j} \nabla_j V(q_j - q_\ell) \left(\frac{p_j}{2} + \frac{p_\ell}{2} \right) = \sum_{\ell > j} J_{j\ell} (q_j - q_\ell) \left(\frac{p_j}{2} + \frac{p_\ell}{2} \right), \quad (6)$$

$$\mathfrak{F}_{\rightarrow j} = \sum_{\ell < j} \nabla_j V(q_j - q_\ell) \left(\frac{p_j}{2} + \frac{p_\ell}{2} \right) = \sum_{\ell < j} J_{j\ell} (q_j - q_\ell) \left(\frac{p_j}{2} + \frac{p_\ell}{2} \right), \quad (7)$$

$$\mathcal{R}_j(t) = \zeta_j (T_j - \langle p_j^2 \rangle). \quad (8)$$

(More details about the derivation of such equations may be found, e.g., in Refs.[14, 21].) $\mathfrak{F}_{j \rightarrow}$ gives the heat current from site j to the forward sites $\ell > j$; $\mathfrak{F}_{\rightarrow j}$ gives the current from the previous sites $\ell < j$. \mathcal{R}_j denotes the energy flux between the j -th site and the j -th reservoir. These models with internal stochastic reservoirs are recurrent, and have been considered in several works [8, 14, 21, 22], usually with the self-consistent condition, which means that the temperatures of the internal reservoirs are chosen such that there is no net energy flux between these internal baths and the system in the steady state, i.e., $\lim_{t \rightarrow \infty} \mathcal{R}_j(t) = 0$, for $j = 2, 3, \dots, N-1$. In other words, in the stationary state with the self-consistent condition we get a heat current across the system supplied only by the external baths at the boundaries with different temperatures. The existence of a steady state in the system with the self-consistent condition (that is, the existence of this suitable choice of internal temperatures) is proven in Ref.[23] and Ref.[8], for the harmonic and anharmonic cases, respectively. In the present paper, we assume that the temperatures $T_1, T_2, \dots, T_{N-1}, T_N$ are arbitrarily given, chosen from a set with lower bound, i.e., there is a T_{min} such that $T_{min} \leq T_1, T_2, \dots, T_N$, for all N . In the Final Remarks section, we recall some physical problems that consider the self-consistent condition.

It is interesting to note the generality of the temperature distribution allowed here. The self-consistent condition, usually assumed with these models with inner reservoirs (and which is considered in the example described in section VI), is related to a specific temperature profile as recalled above, but our approach follows also for many other cases. An interesting problem involving such models of oscillators with inner reservoirs but without the self-consistent condition is presented in Ref.[24].

To proceed, we introduce the phase-space vector $\varphi = (q, p) \in \mathbb{R}^{2N}$, and write the dynamics (2) as

$$d\varphi = -A\varphi dt - \lambda P'(\varphi)dt + \sigma dB, \quad (9)$$

where $A = A^0 + \mathcal{J}$ and σ are $2N \times 2N$ matrices

$$A^0 = \begin{pmatrix} 0 & -I \\ \mathcal{M} & \Gamma \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 0 \\ \mathbb{J} & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2\Gamma\mathcal{T}} \end{pmatrix}, \quad (10)$$

I is the unit $N \times N$ matrix; \mathbb{J} is the $N \times N$ matrix for the interparticle interactions; \mathcal{M} , Γ and \mathcal{T} are the diagonal $N \times N$ matrices, with positive elements: $\mathcal{M}_{jl} = M\delta_{jl}$, $\Gamma_{jl} = \zeta\delta_{jl}$, $\mathcal{T} = T_j\delta_{jl}$. B are independent Brownian motions; $P'(\varphi)$ is a $2N \times 1$ matrix with $P'(\varphi)_j = 0$ for $j = 1, \dots, N$ and

$$P'(\varphi)_i = \frac{dP(\varphi_{i-N})}{d\varphi_{i-N}} \quad \text{for } i = N+1, \dots, 2N. \quad (11)$$

For while, let us use the following index notation: i for indices in the set $\{N+1, N+2, \dots, 2N\}$ (related to momenta coordinates); j for values in the set $\{1, 2, \dots, N\}$ (related to space position coordinates), and k for values in $\{1, 2, \dots, 2N\}$. Throughout the paper, we will omit obvious sums over repeated indices.

As described above (6, 7), the heat flux across the chain is given in terms of two-point functions. Thus, to obtain a mechanism to study heat conduction, we develop an integral representation for the correlation functions, in which a rigorous control is possible (after adjustments such as time discretization) by using standard methods of field theory and equilibrium statistical physics, namely, polymer expansions.

We start the construction of the integral formalism with the solution of the linear ($\lambda = 0$) and decoupled ($\mathcal{J} \equiv 0$) dynamical system. We have

Lemma 1. *The solution $\phi(t)$ of Eq.(9) with $\lambda = 0, \mathcal{J} \equiv 0$, i.e. of equation*

$$d\phi = -A^0\phi dt + \sigma dB,$$

is the Ornstein-Uhlenbeck Gaussian process

$$\phi(t) = e^{-tA^0}\phi(0) + \int_0^t e^{-(t-s)A^0}\sigma dB(s), \quad (12)$$

where, for the case of $\phi(0) = 0$, the covariance of the process evolves as

$$\langle \phi(t)\phi(s) \rangle_0 \equiv \mathcal{C}(t, s) = \begin{cases} e^{-(t-s)A^0}\mathcal{C}(s, s), & t \geq s, \\ \mathcal{C}(t, t)e^{-(s-t)A^{0\dagger}}, & t \leq s, \end{cases} \quad (13)$$

$$\mathcal{C}(t, t) = \int_0^t ds e^{-sA^0}\sigma^2 e^{-sA^{0\dagger}}. \quad (14)$$

Proof. Exercise of stochastic differential equations (see e.g. Ref.[25]). \square

We recall that these solutions may be realized as continues trajectories. Moreover, it follows that

Lemma 2.

$$\lim_{t \rightarrow \infty} \mathcal{C}(t, t) \equiv C = \int_0^\infty ds e^{-sA^0} \sigma^2 e^{-sA^{0\dagger}} = \begin{pmatrix} \frac{\mathcal{T}}{\mathcal{M}} & 0 \\ 0 & \mathcal{T} \end{pmatrix}, \quad (15)$$

and, for any α such that $0 < \alpha < \min \left\{ \frac{\zeta}{2}, \frac{\mathcal{M}}{\mathcal{C}} \right\}$, there is a constant $c < \infty$ such that, for all $t > 0$ and for all N ,

$$\| e^{-tA^0} \| \leq ce^{-t\alpha}. \quad (16)$$

Proof. See Ref.[23] for the proof of Eq.(15) and Ref.[21] for Eq.(16). \square

It is worth to remark that, for the harmonic and decoupled system (i.e., with $\lambda = \mathcal{J} = 0$), each site j is isolated from the other sites and coupled to a single bath at temperature T_j . Then, the expected steady distribution (as $t \rightarrow \infty$) is the related Boltzmann-Gibbs measure, given by the Gaussian measure with measure $d\mu_{\mathcal{C}}$, with the convariance described by Eq.(15), in which each site has the temperature T_j of the bath coupled to it.

Now we use the Girsanov theorem to describe a representation for the correlation functions of $\varphi(t)$, the solution for the complete process (9). We will construct an integral representation for a system with N sites and with time running from 0 to \mathfrak{T} . Later, after the time discretization, we will obtain bounds, valid for all N and \mathfrak{T} , leading to the convergence of the associated polymer expansion.

Theorem 1. *For the correlation functions (6-8) of the crystal chain with reservoirs at each site (1-3), we have the integral representation given by*

$$\langle \varphi_{\ell_1}(t_1) \dots \varphi_{\ell_k}(t_k) \rangle = \int \phi_{\ell_1}(t_1) \dots \phi_{\ell_k}(t_k) \exp[-W(\phi)] d\mu_{\mathcal{C}}, \quad t_1, \dots, t_k \leq \mathfrak{T}, \quad (17)$$

with

$$\begin{aligned} W(\phi) = & \int_0^{\mathfrak{T}} \phi_j(s) \mathcal{J}_{ji}^\dagger \gamma_i^{-1} d\phi_i(s) + \lambda \gamma_i^{-1} P'(\phi)_i(t) d\phi_i(s) + \phi_j(s) \mathcal{J}_{ij}^\dagger \gamma_i^{-1} A_{ik}^0 \phi_k(s) ds + \\ & + \lambda \gamma_i^{-1} P'(\phi)_i(s) A_{ik}^0 \phi_k(s) ds + \frac{1}{2} \phi_{j'}(s) \mathcal{J}_{j'i}^\dagger \gamma_i^{-1} \mathcal{J}_{ij} \phi_j(s) ds + \\ & + \frac{1}{2} \lambda^2 \gamma_i^{-1} (P'(\phi)_i)^2(s) ds + \lambda \gamma_i^{-1} P'(\phi)_i(s) \mathcal{J}_{ij} \phi_j(s) ds, \end{aligned}$$

where ϕ is the solution (given by lemma 1) of the process with $\mathcal{J} \equiv 0$, $\lambda = 0$, and φ is the solution for the complete process (9); the covariance \mathcal{C} is given by equations (13) and (14). The sum over repeated indices i, j, k, \dots is assumed above (and throughout the paper, as already said).

Proof. Girsanov theorem (see e.g. theorem 8.6.8 in Ref.[25]; see also Ref.[26]) gives a measure μ_* for the new process φ in terms of the measure $\mu_{\mathcal{C}}$ associated to previous Ornstein-Uhlenbeck process ϕ with $\mathcal{J} \equiv 0$ and $\lambda = 0$. Precisely, for any measurable set A , it follows that $\mu_*(A) = E_0(1_A Z(\mathfrak{T}))$, where E_0 is the expectation of $\mu_{\mathcal{C}}$, 1_A is the characteristic function, and

$$Z(\mathfrak{T}) = \exp \left(\int_0^{\mathfrak{T}} u \cdot dB - \frac{1}{2} \int_0^{\mathfrak{T}} u^2 ds \right), \quad \text{where } \gamma_i^{1/2} u_i = -\mathcal{J}_{ik} \phi_k - \lambda P'(\phi)_i.$$

The inner products above are in \mathbb{R}^{2N} . Note that, following our previous index convention, u_i is nonzero only for $i \in \{N+1, N+2, \dots, 2N\}$. To apply the Girsanov theorem we must show that $Z(t)$ is a martingale with respect to the σ -algebra generated by $\phi(t)$ and $\mu_{\mathcal{C}}$. To show it, we define the Itô process

$$dX(t) = -\frac{1}{2} u^2 dt + u \cdot dB, \quad X(0) = 0,$$

with u as previously defined. Then, $Z(t) = \exp[X(t)]$ is also an Itô process and

$$dZ(t) = Z(t) u \cdot dB(t) \Rightarrow Z(t) = 1 + \int_0^t Z(s) u(s) \cdot dB(s).$$

As ϕ admits a continuum realization, it follows that $Z(t)$ is bounded and uZ is square-integrable, i.e.,

$$E_0 \left(\int_0^t u^2(s) Z^2(s) ds \right) < \infty.$$

And so, it follows that $Z(t)$ is a martingale (see e.g. corollary 3.2.6 in Ref.[25]).

To conclude, we note that

$$u_i dB_i = \gamma_i^{-1/2} u_i \cdot \gamma_i^{1/2} dB_i = \gamma_i^{-1/2} u_i \cdot (d\phi_i + A_{ik}^0 \phi_k dt) = -\gamma_i^{-1} (\mathcal{J}_{ij} \phi_j + \lambda P'(\phi)_i) (d\phi_i + A_{ik}^0 \phi_k dt),$$

and, finally, we write u^2 in terms of ϕ , see Eq.(18), to obtain the expression as claimed in the theorem. \square

The detailed study of the integral representation above, given by a complicate anharmonic perturbation of a Gaussian measure, seems to be exceedingly difficult. And so, to carry out the rigorous investigation, we try to rewrite the integral representation as an expression in which a more appropriate measure may be considered. But, to establish such new representation, some simplification is necessary. In other words, we propose to follow the investigation in an approximated version of the original problem, derived from the previous formalism as follows (version in which, such strategy of considering a suitable non Gaussian measure is possible).

First, we make an important modification: we introduce an ultraviolet cutoff in the time integral; precisely, we assume discrete times $t = \varepsilon, 2\varepsilon, \dots, \mathfrak{T}$. Second, to avoid unimportant technical difficulties and huge expressions for some subdominant terms (easily controlled in the forthcoming polymer expansion), we simplify the expression for the covariance \mathcal{C} associated to the previous Gaussian process ϕ , given by Eqs.(13, 14). Precisely, we replace the Gaussian measure in the integral representation with such covariance \mathcal{C} by its main part

$$\begin{aligned} d\mu_{\mathcal{C}} &\longrightarrow \exp \left(-\frac{1}{2} \sum_{k,k',t,t'} \phi_k(t) \varepsilon \mathcal{D}^{-1}(t-t') \phi_{k'}(t') \right) \prod_{k,t} d\phi_k(t) / \mathcal{N}, \\ \mathcal{D}^{-1} &\equiv C^{-1} \left(\frac{M}{\zeta} \delta_{t,t'} + c_1 [-\Delta(t,t')] \right), \end{aligned} \quad (18)$$

where \mathcal{N} is the normalization; $-\Delta$ is the discrete Laplacian, $-\Delta(t,s) = 2\delta_{t,s} - \delta_{|t-s|,1}$; and $c_1 = \mathcal{O}(1/\alpha)$ is a small parameter: we assume large $\alpha \equiv \zeta/2$ (i.e., large dissipation), and still take a strong pinning $M = 3\alpha^2$. Details in Appendix A, where we show that this new covariance describes, indeed, the main part of the original Gaussian measure.

From the study of chains of oscillators, it is well known that Fourier's law holds in the harmonic system with internal self-consistent reservoirs [21], but it does not hold anymore if these internal reservoirs are turned off [27]. The scenario is different for the chain with anharmonic on-site potentials: one expects that Fourier's law will be obeyed even without the internal baths. As it is away from our present skill to prove it, to proceed we ignore the possibility of different coupling constants with the internal or external reservoirs and take the same $\zeta_j = \zeta$ for all sites.

Hence, with discrete times and with the simplification of the harmonic covariance, the representation for the correlations (17), e.g. for the two-point function, becomes

$$\begin{aligned} \langle \varphi_{\ell_1}(t) \varphi_{\ell_2}(t) \rangle &\simeq \int \phi_{\ell_1}(t) \phi_{\ell_2}(t) \exp \left\{ - \sum_{s,i,j,\dots} \varepsilon \left[\phi_j(s) \mathcal{J}_{ji}^\dagger \gamma_i^{-1} \frac{[\phi_i(s+\varepsilon) - \phi_i(s)]}{\varepsilon} + \right. \right. \\ &\quad + \gamma_i^{-1} \lambda P'(\phi_{i-N}(s)) \frac{[\phi_i(s+\varepsilon) - \phi_i(s)]}{\varepsilon} + \phi_j(s) \mathcal{J}_{ji}^\dagger \gamma_i^{-1} [M_{i-N} \phi_{i-N}(s) + \zeta \phi_i(s)] + \\ &\quad + \gamma_i^{-1} \lambda_i P'(\phi_{i-N}(s)) M_{i-N} [\phi_{i-N}(s) + \zeta \phi_i(s)] + \frac{1}{2} \phi_{j'}(s) \mathcal{J}_{j'i}^\dagger \gamma_i^{-1} \mathcal{J}_{ij} \phi_j(s) + \\ &\quad + \frac{1}{2} \gamma_i^{-1} \lambda^2 [P'(\phi_{i-N}(s))]^2 + \gamma_i^{-1} \lambda P'(\phi_{i-N}(s)) \mathcal{J}_{ij} \phi_j(s) + \\ &\quad \left. \left. + \frac{1}{2} \phi_k(s) \mathcal{D}_{k,k'}^{-1}(s,s') \phi_{k'}(s') \right] \right\} \prod_{s,k} d\phi_k(s) / \text{normalization}; \end{aligned}$$

moreover, taking $\varepsilon = 1/\zeta$ for simplification, we obtain

$$\begin{aligned} \langle \varphi_{\ell_1}(t_1) \varphi_{\ell_2}(t_2) \rangle &\simeq \int \phi_{\ell_1}(t_1) \phi_{\ell_2}(t_2) \exp \left\{ - \sum_{s,i,j,\dots} \varepsilon \left[\phi_j(s) \mathcal{J}_{ji}^\dagger \gamma_i^{-1} \phi_i(s + \varepsilon) + \right. \right. \\ &+ \frac{\lambda}{\gamma_i} P'(\phi_{i-N}(s)) \phi_i(s + \varepsilon) + \phi_j(s) \mathcal{J}_{ji}^\dagger \frac{M_{i-N}}{\gamma_i} \phi_{i-N}(s) + \\ &+ \frac{\lambda_i}{\gamma_i} P'(\phi_{i-N}(s)) M_{i-N} \phi_{i-N}(s) + \frac{1}{2\gamma_i} \phi_{j'}(s) \mathcal{J}_{j'i}^\dagger \mathcal{J}_{ij} \phi_j(s) + \frac{\lambda^2}{2\gamma_i} [P'(\phi_{i-N}(s))]^2 + \\ &\left. \left. + \frac{\lambda}{\gamma_i} P'(\phi_{i-N}(s)) \mathcal{J}_{ij} \phi_j(s) + \frac{1}{2} \phi_k(s) \mathcal{D}_{k,k'}^{-1}(s, s') \phi_{k'}(s') \right] \right\} \prod_{s,k} d\phi_k(s) / \tilde{\mathcal{N}}, \end{aligned} \quad (19)$$

where, as previously established, $s, s', t, \dots \in \{\varepsilon, 2\varepsilon, \dots, \mathfrak{T}\}$; $j, j' \in \{1, 2, \dots, N\}$; $i, i' \in \{N, N+1, \dots, 2N\}$; $k, k' \in \{1, 2, \dots, 2N\}$; the denominator $\tilde{\mathcal{N}}$ above is the numerator with $\phi_{\ell_1} = \phi_{\ell_2} = 1$, and it was introduced to keep normalized the measure $\exp\{\dots\} \prod d\phi_k(s)$: the original measure given by $Z(\tau) d\mu_C$, which appears before the changes due to time discretization and the simplification of the harmonic part, is normalized.

The main technical achievement of the integral discrete time representation above is that now, as we aimed, a mathematical investigation starting from suitable measures (non Gaussian distributions) will be possible, allowing a profitable perturbative analysis. The polymer expansion, described in the next section, makes it clear.

III. THE POLYMER EXPANSION

We now develop a specific polymer expansion [28–30], suitable for our problem. The technique of polymer (or cluster) expansion is mathematically involved, but it recurrently used in different areas of physics, such as phase transitions in equilibrium statistical mechanics, spectral analysis in field theory, etc. Detailed reviews may be found in textbooks such as Refs.[28, 29, 31]. The existence of a convergent polymer expansion for our present system will allow us to obtain the precise decay of the correlation functions, that is of crucial importance for the study of heat flow in a system with interparticle interactions beyond nearest-neighbor sites [16]. More importantly, a convergent polymer expansion establishes a rigorous support for a perturbative approach for the heat flow investigation, as repeatedly emphasized.

To start the polymer expansion, we need to properly rewrite, reorganize the integral formalism. And so, before describing the technical structures, let us stress the reason of such rearrangement: in these problems involving systems with anharmonic interactions whose behavior is quite different from that observed in related system but with harmonic interactions only (as we have in the heat flow problem for chains of oscillators), a perturbative analysis of the anharmonic model within an integral representation starting with a Gaussian measure (which comes from the harmonic interaction part only) is doomed to failure. For such reason, we are forced to rewrite the integral representation for the correlations in terms of non Gaussian measures: in our case, suitable anharmonic single spin distributions to be described ahead. And so, a perturbative analysis makes sense now: that is, within an integral formalism involving a measure with enough information about the anharmonic potential, we have a suitable starting point, and so, the complete result is reached by adding (now) small corrections to this anharmonic part, corrections which generate a convergent perturbative series.

The notation to be used here and in the following sections is somehow intricate, but is usual in works involving polymer expansion theory, see e.g. Ref.[30] and references there in.

We first consider the term which we name as partition function Z_Λ (as usually named in theory of polymers), that is, the denominator of the two-point function above (19) (i.e. the numerator with $\phi_{\ell_1} = \phi_{\ell_2} = 1$), and rewrite it in terms of polymers. But, instead of an usual single spin distribution, we take as local distributions the measures associated to “cells” of ψ_x , where $\psi_x = (q_x, p_x)$, with

$$q_x = \lambda^{1/3} \phi_{\vec{x}}(x_0) \quad \text{and} \quad p_x = \phi_{\vec{x}+N}(x_0 + \varepsilon),$$

$(x_0, \vec{x}) \in \Lambda = \{\varepsilon, 2\varepsilon, \dots, \mathfrak{T}\} \times \{1, 2, \dots, N\} \subset \mathbb{Z}_*^2 \equiv \varepsilon\mathbb{Z} \times \mathbb{Z}$. We remark that, for clearness in the forthcoming manipulation with polymers, a new notation was introduced above: we replaced the previous time and index notations t, i, j, \dots for $(x_0, \vec{x}) \equiv x$; x_0 for time, and \vec{x} for space. When necessary, we will also split the parts of ψ as q and p . Note that our basic “cell” to be used in the polymer expansion involves ψ_x , in which q and p are in the same site (i.e., in the same space position \vec{x}), but they are nearest neighbors in time.

As the local measure, we define (for $x_0 \neq \varepsilon$ or \mathfrak{T})

$$d\nu(\psi_x) = \frac{e^{-U(\psi_x)}}{C_x} d\psi_x, \quad C_x = \int e^{-U(\psi_x)} d\psi_x, \quad (20)$$

and

$$U_x = \varepsilon \gamma_x^{-1} \left(\frac{1}{2} q_x^6 + q_x^3 p_x + [M + 2\zeta c_1] p_x^2 + \lambda^{-1/3} M q_x^4 \right). \quad (21)$$

To be precise, i.e., to take care of details for the sites at the ends, for $x_0 = \varepsilon$ and $x_0 = \mathfrak{T}$ we define

$$\begin{aligned} d\nu(\psi_x(x_0 = \varepsilon)) &= \exp \left\{ -U_x(x_0 = \varepsilon) - \varepsilon \gamma_x^{-1} [M + 2\zeta c_1] p_x^2(x_0 = \varepsilon) \right\} \times \\ &\quad \times d\psi_x(x_0 = \varepsilon) dp_{\bar{x}}(x_0 = \varepsilon) / \text{normalization}, \\ d\nu(\psi_x(x_0 = \mathfrak{T})) &= \exp \left\{ -\varepsilon \gamma_x^{-1} \left[q_{\bar{x}}^6(x_0 = \mathfrak{T}) + \lambda^{-1/3} M q_{\bar{x}}^4(x_0 = \mathfrak{T}) \right] \right\} \times \\ &\quad \times dq_{\bar{x}}(x_0 = \mathfrak{T}) / \text{normalization}. \end{aligned} \quad (22)$$

Hence, see Eq.(19), the partition function Z_Λ is given by

$$Z_\Lambda = C^\Lambda \prod_{x \in \Lambda} \left(\int d\nu(\psi_x) \prod_{\{x,y\} \subset \Lambda} e^{G_{xy}(\psi_x, \psi_y)} \right), \quad (23)$$

where, $C^\Lambda \equiv \prod_{x \in \Lambda} C_x$, and $G_{xy} = -\sum_{k=1}^6 G_{xy}^{(k)}$,

$$\begin{aligned} G_{xy}^{(1)} &= A_{xy}^{(1)} q_x p_y = \varepsilon \gamma_x^{-1} J_{\bar{x}\bar{y}} (1 - \delta_{\bar{x}\bar{y}}) \lambda^{-1/3} \delta_{x_0, y_0} q_x p_y, \quad G_{xy}^{(2)} = A_{xy}^{(2)} q_x q_y = \varepsilon \gamma_x^{-1} J_{\bar{x}\bar{y}} (1 - \delta_{\bar{x}\bar{y}}) \lambda^{-2/3} M \delta_{x_0, y_0} q_x q_y, \\ G_{xy}^{(3)} &= A_{xy}^{(3)} q_x q_y = \varepsilon \sum_{\substack{\bar{k} \\ \bar{k} \neq \bar{x}, \bar{y}}} \frac{\gamma_{\bar{k}}^{-1} \lambda^{-2/3}}{4} J_{\bar{x}\bar{k}} J_{\bar{k}\bar{y}} \delta_{x_0, y_0} q_x q_y, \quad G_{xy}^{(4)} = A_{xy}^{(4)} q_x^3 q_y = \varepsilon \gamma_x^{-1} J_{\bar{x}\bar{y}} (1 - \delta_{\bar{x}\bar{y}}) \lambda^{-1/3} \delta_{x_0, y_0} q_x^3 q_y, \\ G_{xy}^{(5)} &= A_{xy}^{(5)} q_x q_y = 2\varepsilon \gamma_x^{-1} \lambda^{-2/3} M \delta_{\bar{x}\bar{y}} [\delta_{x_0, y_0} - c_1 (\Delta(x_0, y_0))] q_x q_y, \quad G_{xy}^{(6)} = A_{xy}^{(6)} p_x p_y = -\varepsilon \gamma_x^{-1} \zeta c_1 \delta_{|x_0 - y_0|, \varepsilon} \delta_{\bar{x}\bar{y}} p_x p_y. \end{aligned}$$

We rewrite the partition function as

$$Z_\Lambda = C^\Lambda \prod_{x \in \Lambda} \int d\nu(\psi_x) \prod_{\{x,y\} \subset \Lambda} \left(e^{G_{xy}(\psi_x, \psi_y)} - 1 + 1 \right) = C^\Lambda \Xi_\Lambda, \quad (24)$$

with

$$\Xi_\Lambda = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{R_1, \dots, R_n \subset \Lambda \\ R_i \cap R_j = \emptyset, |R_i| \geq 2}} \rho(R_1) \dots \rho(R_n), \quad (25)$$

where $R_1, \dots, R_n \subset \Lambda$ is a collection of subsets of Λ with cardinality greater than 1, with associated activities $\rho(R)$ given by

$$\rho(R) = \prod_{x \in R} \int d\nu(\psi_x) \sum_{g \in G_R} \prod_{\{x,y\} \subset \Lambda} (e^{G_{xy}(\psi_x, \psi_y)} - 1), \quad (26)$$

where $\sum_{g \in G_R}$ is the sum over the connected graphs on the set R . Given a finite set A , we define a graph g in A as a collection $\{\gamma_1, \dots, \gamma_m\}$ of distinct pairs of A , i.e., $\gamma_i = \{x_i, y_i\} \subset A$ with $x_i \neq y_i$. A graph $g = \{\gamma_1, \dots, \gamma_m\}$ in A is connected if for any B, C of subsets of A such that $B \cup C = A$ and $B \cap C = \emptyset$, there is a $\gamma_i \in g$ such that $\gamma_i \cap B \neq \emptyset$ and $\gamma_i \cap C \neq \emptyset$. The pairs γ_i are called links of the graph. We denote by $|g|$ the number of links in g .

From the standard polymer theory [28, 29], it follows that we can expand $\log \Xi_\Lambda$ as

$$\log \Xi_\Lambda = \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{R_1, \dots, R_n \subset \Lambda \\ |R_i| \geq 2}} \phi^T(R_1, \dots, R_n) \rho(R_1) \dots \rho(R_n), \quad (27)$$

with

$$\phi^T(R_1, \dots, R_n) = \begin{cases} 1 & , \text{ if } n = 1, \\ \sum_{f \in g(R_1, \dots, R_n)} (-1)^{|f|} & , \text{ if } n \geq 2 \text{ and } g(R_1, \dots, R_n) \in G_n, \\ 0 & , \text{ if } g(R_1, \dots, R_n) \notin G_n, \end{cases}$$

where G_n above denotes the set of the connected graphs on $\{1, \dots, n\}$, and $g(R_1, \dots, R_n)$ denotes the graph in $\{1, 2, \dots, n\}$ which has the link $\{i, j\}$ if and only if $R_i \cap R_j \neq \emptyset$.

The connection between the polymer expansion and a perturbative series is clear from the expressions (24, 25, 27) above. To make it explicit and transparent, note that if we write G_{xy} as βG_{xy} , then $\rho = \mathcal{O}(\beta)$ (it involves $\exp[\beta G_{xy}] - 1$), and the polymer expansions above give us a power series in β .

A sufficient condition for the convergence of the polymer series (27), uniformly in Λ , is given by the well known result due to Kotecký and Preiss:

Lemma 3. *If there is $a > 0$ such that*

$$\sup_{x \in \Lambda} \sum_{x \in R \subset \Lambda} |\rho(R)| e^{a|R|} < a, \quad (28)$$

then

$$|\log \Xi_\Lambda| \leq \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{R_1, \dots, R_n \subset \Lambda \\ |R_i| \geq 2}} |\phi^T(R_1, \dots, R_n) \rho(R_1) \dots \rho(R_n)| \leq E_a |\Lambda|, \quad (29)$$

where E_a does not depend on Λ .

Proof. See Refs.[31–33]. □

In relation to the two-point correlation (19), we can rewrite it as

$$S_2(x_1; x_2) = \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \log \tilde{\Xi}_\Lambda(\alpha_1, \alpha_2) \Big|_{\alpha=0}, \quad (30)$$

with

$$\tilde{\Xi}_\Lambda(\alpha_1, \alpha_2) = \prod_{x \in \Lambda} \int d\nu(\psi_x) e^{G_{xy}(\psi_x, \psi_y)} (1 + \alpha_1 \psi_{x_1}^{(c)}) (1 + \alpha_2 \psi_{x_2}^{(c)}), \quad (31)$$

where $\psi_x^{(c)} = q_x(x_0)$ or $p_x(x_0)$. Note that $\tilde{\Xi}_\Lambda(\alpha_1 = 0, \alpha_2 = 0) = \Xi_\Lambda$. Again, we expand $\tilde{\Xi}_\Lambda(\alpha_1, \alpha_2)$ in terms of polymers. For any $R \subset \Lambda$, we denote by I_R the subset (possibly empty) of $\{1, 2\}$ such that $i \in I_R$ iff $x_i \in R$, where $i = 1, 2$. We have

$$\Xi_\Lambda(\alpha_1, \alpha_2) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{R_1, \dots, R_n \subset \Lambda \\ R_i \cap R_j = \emptyset \text{ } |R_i| \geq 1}} \tilde{\rho}(R_1, \alpha) \dots \tilde{\rho}(R_n, \alpha), \quad (32)$$

where

$$\tilde{\rho}(R, \alpha) = \begin{cases} \prod_{x \in R} \int d\nu(\psi_x) \prod_{i \in I_R} (1 + \alpha_i \psi_{x_i}^{(c)}) \sum_{g \in G_R} \prod_{\{x, y\} \in g} (e^{G_{xy}(\psi_x, \psi_y)} - 1) & , \text{ for } |R| \geq 2, \\ \prod_{x \in R} \int d\nu(\psi_x) \prod_{i \in I_R} \alpha_i \psi_{x_i}^{(c)} & , \text{ for } I_R \neq \emptyset, |R| = 1, \\ 0 & , \text{ for } I_R = \emptyset, |R| = 1. \end{cases} \quad (33)$$

It is important to note that the one-body polymers $R = \{x\}$ can also contribute to the partition function (32), but only if $x = x_i$ for some $i \in \{1, 2\}$.

If we take the log of (32) and note that only the terms proportional to $\alpha_1 \alpha_2$ give a non vanishing contribution to the two-point truncated correlation function, we obtain

$$S_2(x_1; x_2) = \sum_{n \geq 1} \frac{1}{n!} \sum_{i_1, i_2=1}^n \sum_{\substack{R_1, \dots, R_n \subset \Lambda, \\ R_{i_1} \ni x_1, R_{i_2} \ni x_2 \\ |R_j| \geq 2}} \phi^T(R_1, \dots, R_n) \tilde{\rho}(R_1) \dots \tilde{\rho}(R_n), \quad (34)$$

where

$$\begin{aligned} \tilde{\rho}(R_i) = & \prod_{x \in R_i} \int d\nu(\psi_x) \left[\left(\psi_{x_1}^{(c)} \right)^{\beta_i^1} + \beta_i^1 l_1 \right] \left[\left(\psi_{x_2}^{(c)} \right)^{\beta_i^2} + \beta_i^2 l_2 \right] \times \\ & \times \sum_{g \in G_R} \prod_{\{x,y\} \in g} (e^{G_{xy}(\psi_x, \psi_y)} - 1), \end{aligned} \quad (35)$$

with $\beta_i^j = 0$ if $i \neq j$, or 1 if $i = j$ and $l_k = \int d\nu(\psi) \psi_{x_k}^{(c)}$. Due to the fact that R_1, \dots, R_n must be connected, the one-body polymers are absorbed in the activity of the many body polymers in the terms proportional to l above. And so, each 1-body polymer (if any) is always contained in, at least, one many-body polymer.

IV. CONVERGENCE OF THE POLYMER EXPANSION

We will describe, in detail, the case in which the interparticle potential $J_{\vec{x}, \vec{y}}$ has an integrable polynomial decay (cases with exponential decay or with finite range can be treated in a similar, but easier, way). Precisely, we assume here that, for $\vec{x} \neq \vec{y}$,

$$\frac{J'}{|\vec{x} - \vec{y}|^p} \leq J_{\vec{x}, \vec{y}} \leq \frac{J}{|\vec{x} - \vec{y}|^p}, \quad (36)$$

where J, J' are real constants and $p \geq 1 + \epsilon$ with $\epsilon > 0$.

In what follows, we assume the regime of large dissipation, that means ζ large (and so, large $\alpha = \zeta/2$ and large harmonic pinning constant $M = 3\alpha^2$, see appendix A), and, more importantly, we also assume the regime of large anharmonicity, i.e., at the end we take λ as large as necessary.

Our strategy is to prove that the Kotecký-Preiss's condition (28) is satisfied by our specific polymer expansion. But, before carrying out any computation, in order to control the exaggerated number of graphs that appears in the expression for the activity $\rho(R)$, we recall an important and well known result (to be used ahead), namely, the Brydges-Battle-Federbush tree graph inequality:

Lemma 4. *Let R be a finite set with cardinality $|R|$ and let $\{V_{xy} : \{x, y\} \subset R\}$ be a set with $|R|(|R| - 1)/2$ real numbers (precisely, $\{x, y\}$ are unordered pairs in R). Suppose that there exist $|R|$ positive numbers V_x (with $x \in R$) such that, for any subset $S \subset R$,*

$$\left[\sum_{x \in S} V_x + \sum_{\{x, y\} \in S} V_{xy} \right] \geq 0. \quad (37)$$

Then

$$\left| \sum_{g \in G_R} \prod_{\{x, y\} \in g} (e^{-V_{xy}} - 1) \right| \leq e^{\sum_{x \in R} V_x} \sum_{\tau \in T_R} \prod_{\{x, y\} \in \tau} |V_{xy}|, \quad (38)$$

where T_R denotes the set of the tree graphs on R .

Proof. See Refs.[31, 34]. □

To ensure the Kotecký-Preiss's condition (28), we will bound the factor

$$\varepsilon_n(z, z') = \sum_{\substack{R \subset \Lambda: |R|=n \\ z, z' \in R}} |\rho(R)|, \quad (39)$$

and show that $\varepsilon_n(z, z') \leq [f(J, \lambda^{-1}, c_1)]^n$, where $f(J, \lambda c_1) \rightarrow 0$ as $\lambda^{-1}, J, c_1 \rightarrow 0$.

Note that for $\varphi, \psi \in \mathbb{R}$, $x \in \Lambda$, we have

$$\begin{aligned} 2|\varphi||\psi| &\leq \varphi^2 + \psi^2, \quad |\varphi|^a |\psi|^b \leq |\varphi|^{a+b} + |\psi|^{a+b} \quad \text{for } a, b \geq 0, \\ \sum_{y \in \Lambda} \delta_{|x_0 - y_0|, \varepsilon} \delta_{\vec{x}\vec{y}} &\leq \sum_{y_0 \in \mathbb{Z}} \delta_{|x_0 - y_0|, \varepsilon} \leq 2, \quad \sum_{y \in \Lambda} J_{\vec{x}\vec{y}} \delta_{x_0, y_0} \leq \sup_{\vec{x} \in \mathbb{Z}} \sum_{\vec{y} \in \mathbb{Z}} |J_{\vec{x}\vec{y}}| \leq J_M, \\ \sum_{\substack{k \in \mathbb{Z} \\ \vec{k} \neq \vec{x}, \vec{y}}} J_{\vec{x}\vec{k}} J_{\vec{k}\vec{y}} &\leq \sum_{\substack{k \in \mathbb{Z} \\ \vec{k} \neq \vec{x}, \vec{y}}} \frac{J}{|\vec{x} - \vec{k}|^p} \frac{J}{|\vec{k} - \vec{y}|^p} \leq \frac{J^2 \mathcal{O}(1)}{|\vec{x} - \vec{y}|^p} (1 - \delta_{\vec{x}, \vec{y}}). \end{aligned}$$

By using the bound $\gamma_x^{-1} \leq \gamma^{-1} = (2\zeta T_{\min})^{-1}$, where $T_{\min} = \min_x \{T_x\}$, we obtain

$$\begin{aligned}
\left| \sum_{\{x,y\} \subset R} G_{xy}^{(k)} \right| &\leq \sum_{\{x,y\} \subset R} |G_{xy}^{(k)}| \\
&\leq \sum_{\{x,y\} \subset R} |A_{xy}^{(k)}| |\varphi_x|^a |\psi_y|^b \leq \sum_{\{x,y\} \subset R} |A_{xy}^{(k)}| (|\varphi_x|^{a+b} + |\psi_y|^{a+b}) \\
&\leq \sum_{x \in R} (|\varphi_x|^{a+b} + |\psi_x|^{a+b}) \sum_{y \in R} |A_{xy}^{(k)}|.
\end{aligned} \tag{40}$$

Hence,

$$\begin{aligned}
\left| \sum_{x,y \in \Lambda} G_{xy}^{(1)} \right| &\leq \sum_{x \in R} \varepsilon \gamma^{-1} J_M \lambda^{-1/3} \frac{q_x^2 + p_x^2}{2}, & \left| \sum_{x,y \in \Lambda} G_{xy}^{(2)} \right| &\leq \sum_{x \in R} \varepsilon M \gamma^{-1} J_M \lambda^{-2/3} q_x^4, \\
\left| \sum_{x,y \in \Lambda} G_{xy}^{(3)} \right| &\leq \sum_{x \in R} \varepsilon \gamma^{-1} J_M^2 \lambda^{-2/3} \mathcal{O}(1) q_x^2, & \left| \sum_{x,y \in \Lambda} G_{xy}^{(4)} \right| &\leq \sum_{x \in R} 2\varepsilon \gamma^{-1} J_M^2 \lambda^{-1/3} q_x^4, \\
\left| \sum_{x,y \in \Lambda} G_{xy}^{(5)} \right| &\leq \sum_{x \in R} \varepsilon M \gamma^{-1} J_M^2 \lambda^{-2/3} (1 + 4c_1) \frac{q_x^2}{2}, & \left| \sum_{x,y \in \Lambda} G_{xy}^{(6)} \right| &\leq \sum_{x \in R} 2\varepsilon \gamma^{-1} \zeta c_1 p_x^2.
\end{aligned}$$

And so,

$$\left| \sum_{\{x,y\} \subset R} G_{xy} \right| \leq \sum_{x \in R} \mathcal{P}(q_x, p_x), \tag{41}$$

where $\mathcal{P}(q_x, p_x)$ is a polynomial of degree 4 in q_x and 2 in p_x , and it is bounded from below. Hence, there are constants C_1, C_2 and C_3 depending on $\varepsilon, \lambda, J, M, \gamma$ such that $\mathcal{P}(q_x, p_x) \leq C_1 q_x^4 + (C_2 + 2\varepsilon \gamma^{-1} \zeta c_1) p_x^2 + C_3$. By using the Brydges-Battle-Federbush tree graph inequality, we get

$$\left| \sum_{g \in G_R} \prod_{\{x,y\} \in g} (e^{G_{xy}} - 1) \right| \leq \prod_{x \in R} e^{\mathcal{P}(q_x, p_x)} \sum_{\tau \in T_R} \prod_{\{x,y\} \in \tau} |G_{xy}|. \tag{42}$$

Consequently,

$$\begin{aligned}
\sum_{\substack{R \subset \Lambda: |R| \geq 2 \\ z, z' \in R}} |\rho(R)| e^{|R|} &= \sum_{n \geq 2} e^n \sum_{\substack{R \subset \Lambda: |R|=n \\ z, z' \in R}} |\rho(R)| \\
&= \sum_{n \geq 2} \frac{e^n}{(n-2)!} \sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} |\rho(R = \{x_1, \dots, x_n\})| \\
&\leq \sum_{n \geq 2} \frac{e^n}{(n-2)!} \sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} \int \prod_{i=1}^n d\nu(\psi_{x_i}) e^{\mathcal{P}(q_x, p_x)} \sum_{\tau \in T_n} \prod_{\{i,j\} \in \tau} |G_{x_i x_j}|.
\end{aligned}$$

Recall now that $|\tau| = n - 1$. Hence, fixing $\tau \in T_n$, we have

$$\begin{aligned}
\prod_{\{i,j\} \in \tau} |G_{x_i x_j}| &\leq \prod_{\{i,j\} \in \tau} \sum_{s=1}^6 |A_{x_i x_j}^{(s)}| |q_{x_i}^{a_s^{(i)}}| |p_{x_i}^{b_s^{(i)}}| |q_{x_i}^{a_s^{(j)}}| |p_{x_j}^{b_s^{(j)}}| \\
&\leq \sum_{\{i,j\} \in \tau} \sum_{s_{ij}=1}^6 \prod_{k=1}^n |q_{x_k}|^{n_k(s)} |p_{x_k}|^{m_k(s)} \prod_{\{i,j\} \in \tau} |A_{x_i x_j}^{(s_{ij})}|,
\end{aligned} \tag{43}$$

where $\{s_{ij}\}_{\{i,j\} \in \tau}$ is a sequence of possible choices of s 's (from 1 to 6), for each line $\{i, j\} \in \tau$. The exponents $n_k(s)$ and $m_k(s)$ depend on such sequence, and depend also on the exponents a_s and b_s of q_{x_i} and p_{x_i} in each $G_{xy}^{(s)}$. In any case, we have the bounds $0 \leq n_k(s) \leq d_k \cdot \max\{a_s\} \leq 3d_k$ and $0 \leq m_k(s) \leq d_k \cdot \max\{b_s\} \leq d_k$, where $\{d_k\}_{k=1}^n$ are the incidence indices of the tree $\tau \in T_n$, with $1 \leq d_k \leq n-1$ and $\sum_{k=1}^n d_k = 2n-2$. Then

$$\begin{aligned}
& \sum_{\substack{R \subset \Lambda: |R| \geq 2 \\ z, z' \in R}} |\rho(R)| e^{|R|} \leq \sum_{n \geq 2} \frac{e^n}{(n-2)!} \sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} \int \prod_{i=1}^n d\nu(\psi_{x_i}) e^{\mathcal{P}(q_{x_i}, p_{x_i})} \times \\
& \times \sum_{\tau \in T_n} \sum_{\{i,j\} \in \tau} \sum_{s_{ij}=1}^6 \prod_{k=1}^n |q_{x_k}|^{n_k(s)} |p_{x_k}|^{m_k(s)} \prod_{\{i,j\} \in \tau} |A_{x_i x_j}^{(s_{ij})}| \leq \\
& \leq \sum_{n \geq 2} \frac{e^n}{(n-2)!} \sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} \sum_{\tau \in T_n} \sum_{\{i,j\} \in \tau} \sum_{s_{ij}=1}^6 \\
& \prod_{k=1}^n \left(\int d\nu(\psi_{x_k}) e^{\mathcal{P}(q_{x_i}, p_{x_i})} |q_{x_k}|^{n_k(s)} |p_{x_k}|^{m_k(s)} \right) \prod_{\{i,j\} \in \tau} |A_{x_i x_j}^{(s_{ij})}|. \tag{44}
\end{aligned}$$

Lemma 5. $\forall \alpha, \beta > 0$, $\alpha, \beta \in \mathbb{R}$, $C_1 < \frac{\varepsilon \gamma^{-1}}{6}$ and $C_2 < \frac{\varepsilon \gamma^{-1}}{4}$, we have

$$\int d\nu(\psi) |q|^\alpha |p|^\beta e^{\mathcal{P}(q,p)} \leq \frac{\gamma^{-1/2} e^{K_5 - K_2} K_1^{1/6} (1+c_1)^{1/2}}{3K_3^{\frac{1+\alpha}{6}} K_4^{\frac{1+\beta}{2}}} \Gamma\left(\frac{1+\alpha}{6}\right) \Gamma\left(\frac{1+\beta}{2}\right),$$

where K_1, \dots, K_5 , are constants such that, $\forall q, p \in \mathbb{R}$,

$$\gamma^{-1} \left\{ \left[\frac{1}{2} - \frac{1}{4(M+2\zeta c_1)} \right] q^6 + \lambda^{-1/3} M q^4 \right\} \leq K_1 q^6 + K_2,$$

and

$$U(q, p) - \mathcal{P}(q, p) \geq K_3 q^6 + K_4 p^2 + K_5.$$

Proof. We have

$$\begin{aligned}
U(q, p) &= \varepsilon \gamma^{-1} \left(\frac{1}{2} q^6 + q^3 p + (M + 2\zeta c_1) p^2 + \lambda^{-1/3} M q^4 \right) \\
&= \varepsilon \gamma^{-1} \left\{ \left[\frac{q^3}{2(M+2\zeta c_1)^{1/2}} + (M+2\zeta c_1)^{1/2} p \right]^2 - \frac{q^6}{4(M+2\zeta c_1)} + \frac{q^6}{2} + \lambda^{-1/3} M q^4 \right\} \\
&\leq \varepsilon \gamma^{-1} \left[\frac{q^3}{2(M+2\zeta c_1)^{1/2}} + (M+2\zeta c_1)^{1/2} p \right]^2 + K_1 q^6 + K_2. \tag{45}
\end{aligned}$$

It also follows that

$$\begin{aligned}
U(q, p) &= \varepsilon \gamma^{-1} \left(\frac{1}{2} q^6 + q^3 p + (M + 2\zeta c_1) p^2 + \lambda^{-1/3} M q^4 \right) \\
&\geq \varepsilon \gamma^{-1} \left(\frac{1}{2} q^6 + q^3 p + (1 + 2\zeta c_1) p^2 + \lambda^{-1/3} M q^4 \right) \\
&= \varepsilon \gamma^{-1} \left[\frac{q^6}{6} + \left(\frac{1}{\sqrt{3}} q^3 + \frac{\sqrt{3}}{2} p \right)^2 + \frac{p^2}{4} + 2\zeta c_1 p^2 \right] \geq \varepsilon \gamma^{-1} \left[\frac{q^6}{6} + \frac{p^2}{4} + 2\zeta c_1 p^2 \right]. \tag{46}
\end{aligned}$$

Hence,

$$\begin{aligned}
U(q, p) - \mathcal{P}(q, p) &\geq \left(\frac{\varepsilon \gamma^{-1}}{6} - C_1 \right) q^6 + \varepsilon \lambda^{-1/3} M q^4 + \left(\frac{\varepsilon \gamma^{-1}}{4} - C_2 \right) p^2 - C_3 \\
&\geq K_3 q^6 + K_4 p^2 + K_5, \tag{47}
\end{aligned}$$

with $K_3, K_4 > 0$, as $C_1 < \frac{\varepsilon\gamma^{-1}}{6}$ and $C_2 < \frac{\varepsilon\gamma^{-1}}{4}$. From the definition of the s.s.d., we have

$$\int d\nu(\psi)|q|^\alpha|p|^\beta e^{\mathcal{P}(q,p)} = \frac{1}{C_I} \int d\psi|q|^\alpha|p|^\beta e^{\mathcal{P}(q,p)-U(q,p)}.$$

And so,

$$\begin{aligned} C_I &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-U(q,p)} dp dq \\ &\geq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ -\varepsilon\gamma^{-1} \left[\frac{q^3}{2(M+2\zeta c_1)^{1/2}} + (M+2\zeta c_1)^{1/2} p \right]^2 - K_1 q^6 - K_2 \right\} dp dq \\ &= \int_{-\infty}^{\infty} \sqrt{\frac{\pi\gamma}{\varepsilon(M+2\zeta c_1)}} e^{-K_1 q^6 - K_2} dq = 2 \sqrt{\frac{\pi\gamma}{\varepsilon(M+2\zeta c_1)}} e^{-K_2} K_1^{-1/6} \\ &\geq \sqrt{\frac{\gamma}{\varepsilon(M+2\zeta c_1)}} e^{-K_2} K_1^{-1/6} \Gamma\left(\frac{7}{6}\right), \end{aligned}$$

where the last inequality comes from $2\sqrt{\pi}\Gamma\left(\frac{7}{6}\right) \approx 3.3 > 1$. We still have

$$\begin{aligned} \int d\psi|q|^\alpha|p|^\beta e^{\mathcal{P}(q,p)-U(q,p)} &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{K_3 q^6 + K_4 p^2 + K_5} dp dq = \\ &= \frac{1}{3} e^{-K_5} K_3^{-\frac{1+\alpha}{6}} K_4^{-\frac{1+\beta}{2}} \Gamma\left(\frac{\alpha+1}{6}\right) \Gamma\left(\frac{\beta+1}{2}\right). \end{aligned}$$

And the lemma's proof follows from these two bounds. \square

Using the fact that for large x, y , e.g. $x, y > 1$, there exists a constant c , such that $\Gamma(x)\Gamma(y) \leq c\Gamma(x+y-1)$, it follows that

$$\Gamma\left(\frac{1+n(k)}{6}\right) \Gamma\left(\frac{1+m(k)}{2}\right) \leq K_6 \Gamma\left(\frac{1+n(k)}{6} + \frac{1+m(k)}{2} - 1\right) \leq K_6 \Gamma(d_k),$$

for a positive constant K_6 , chosen to take care of possible small values of x, y in $\Gamma(x)\Gamma(y)$. Let $\tilde{K}_3 = \min\{1, K_3^{1/2}\}$ and $\tilde{K}_4 = \min\{1, K_4^{1/2}\}$. So, $K_3^{\frac{n(k)}{6}} \geq \tilde{K}_3^{d_k}$ and $K_4^{\frac{m(k)}{2}} \geq \tilde{K}_4^{d_k}$. Using this with the lemma, we obtain

$$\begin{aligned} \sum_{\substack{R \subset \Lambda: |R| \geq 2 \\ z, z' \in R}} |\rho(R)| e^{|R|} &\leq \sum_{n \geq 2} \frac{e^n}{(n-2)!} \sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} \sum_{\tau \in T_n} \sum_{\{i,j\} \in \tau} \sum_{s_{ij}=1}^6 \\ &\quad \times \prod_{k=1}^n \left[\frac{\varepsilon^{1/2} \gamma^{-1/2} e^{K_5 - K_2} K_1^{1/6} (M+2\zeta c_1)^{1/2}}{3 K_3^{\frac{1}{6}} K_4^{\frac{1}{2}}} \tilde{K}_3^{-d_k} \tilde{K}_4^{-d_k} K_6 \Gamma(d_k) \right] \\ &\quad \times \prod_{\{i,j\} \in \tau} |A_{x_i x_j}^{(s_{ij})}| \\ &\leq \sum_{n \geq 2} \frac{e^n}{(n-2)!} \left(\frac{\varepsilon^{1/2} \gamma^{-1/2} e^{K_5 - K_2} K_1^{1/6} (M+2\zeta c_1)^{1/2} K_6}{3 K_3^{\frac{1}{6}} K_4^{\frac{1}{2}}} \right)^n \tilde{K}_3^{-2n+2} \tilde{K}_4^{-2n+2} \\ &\quad \times \sum_{\tau \in T_n} \sum_{\{i,j\} \in \tau} \sum_{s_{ij}=1}^6 \prod_{k=1}^n \Gamma(d_k) \sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} \prod_{\{i,j\} \in \tau} |A_{x_i x_j}^{(s_{ij})}|, \end{aligned} \tag{48}$$

where we used $\prod_{k=1}^n \phi^{d_k} = \phi^{d_1+d_2+\dots+d_n} = \phi^{2n-2}$.

We note that for any $\tau \in T_n$, there is a unique path $\bar{\tau}$ in τ which joins vertex 1 to vertex 2. Fixing $\tau \in T_n$, let be $\bar{\tau} \equiv \{1, i_1\}, \{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{k-1}, i_k\}, \{i_k, 2\}$ and $I_\tau \equiv \{1, i_1, i_2, \dots, i_k, 2\}$ the subset of $\{1, 2, 3, \dots, n\}$ whose elements are the vertices of

the path $\bar{\tau}$. Hence, $|\tau| = n - 1$, $|\bar{\tau}| = k + 1$ and $|\tau \setminus \bar{\tau}| = n - k - 2$.

From the definitions of $|A_{xy}^{(s)}| : s = 1, \dots, 6$, we see that all the terms vanish if $|x_0 - y_0| > \varepsilon$. Hence, fixing $\tau \in T_n$, if $\exists \{i, j\} \in \tau$ such that $|(x_i)_0 - (x_j)_0| > \varepsilon$ we have $|A_{x_i x_j}^{(s)}| = 0 \forall s = 1, \dots, (c + 1)$, and so, this tree τ does not contribute to the sum (48). Then, given $|(x_1)_0 - (x_2)_0|$, as $|\bar{\tau}| = k + 1$, if $|(x_1)_0 - (x_2)_0| > \varepsilon(k + 1)$ then $\exists \{i, j\} \in \bar{\tau}$ such that $|(x_i)_0 - (x_j)_0| > \varepsilon$, and so, τ does not contribute to (48). As $\bar{\tau} \subset \tau$ we have $n - 1 \geq k + 1$. Therefore, any tree $\tau \in T_n$ such that $|(x_1)_0 - (x_2)_0| > \varepsilon(n - 1) \geq \varepsilon(k + 1)$ does not contribute to (48), in other words, $\rho(R)$ vanishes if $|(x_1)_0 - (x_2)_0| > \varepsilon(|R| - 1)$, i.e., if $|R| < \frac{|(x_1)_0 - (x_2)_0|}{\varepsilon} + 1$.

We define $\mathcal{N}' \equiv \max \left\{ \frac{|z_0 - z'_0|}{\varepsilon} + 1, 2 \right\}$, and we have

$$\sum_{\substack{R \subset \Lambda : |R| \geq 2 \\ z, z' \in R}} |\rho(R)| e^{|R|} = \sum_{\substack{R \subset \Lambda : |R| \geq \mathcal{N}' \\ z, z' \in R}} |\rho(R)| e^{|R|}. \quad (49)$$

Now, we note that

$$\delta_{\vec{x}, \vec{y}} \delta_{|x_0 - y_0|, \varepsilon} \leq \delta_{\vec{x}, \vec{y}} e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1}, \quad \delta_{x_0, y_0} \leq e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1}.$$

Then,

$$\begin{aligned} |A_{xy}^{(1)}| &= \varepsilon \gamma^{-1} J_{\vec{x}\vec{y}} (1 - \delta_{\vec{x}\vec{y}}) \lambda^{-1/3} \delta_{x_0, y_0} \leq \\ &\leq \varepsilon \gamma^{-1} J \lambda^{-1/3} e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} \right) = A_1 e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} \right), \\ |A_{xy}^{(2)}| &= \varepsilon \gamma^{-1} J_{\vec{x}\vec{y}} (1 - \delta_{\vec{x}\vec{y}}) \lambda^{-2/3} M \delta_{x_0, y_0} \leq \\ &\leq \varepsilon \gamma^{-1} J \lambda^{-2/3} M e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} \right) = A_2 e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} \right), \\ |A_{xy}^{(3)}| &= \sum_{\substack{\vec{k} \\ \vec{k} \neq \vec{x}, \vec{y}}} \frac{\varepsilon \gamma^{-1} \lambda^{-2/3}}{4} J_{\vec{x}\vec{k}} J_{\vec{k}\vec{y}} (1 - \delta_{\vec{x}\vec{y}}) \delta_{x_0, y_0} \leq \\ &\leq \frac{\varepsilon \gamma^{-1} \lambda^{-2/3}}{4} J J_M \mathcal{O}(1) e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} \right) = A_3 e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} \right), \\ |A_{xy}^{(4)}| &= \varepsilon \gamma^{-1} J_{\vec{x}\vec{y}} (1 - \delta_{\vec{x}\vec{y}}) \lambda^{-1/3} \delta_{x_0, y_0} \leq \\ &\leq \varepsilon \gamma^{-1} J \lambda^{-1/3} e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} \right) = A_4 e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} \right), \\ |A_{xy}^{(5)}| &= 2\varepsilon \gamma^{-1} \lambda^{-2/3} M \delta_{\vec{x}\vec{y}} |\delta_{x_0, y_0} - c_1 \Delta(x_0, y_0)| \leq \\ &\leq 2\varepsilon \gamma^{-1} \lambda^{-2/3} M (1 + 3c_1) e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \delta_{\vec{x}\vec{y}} = A_5 e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \delta_{\vec{x}\vec{y}}, \\ |A_{xy}^{(6)}| &= \varepsilon \gamma^{-1} \zeta c_1 \delta_{|x_0 - y_0|, 1} \delta_{\vec{x}\vec{y}} \leq \varepsilon \gamma^{-1} \zeta c_1 e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \delta_{\vec{x}\vec{y}} = A_6 e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \delta_{\vec{x}\vec{y}}. \end{aligned}$$

Hence, for $s = 1, \dots, 6$,

$$|A_{xy}^{(s)}| \leq A_s e^{-\frac{|x_0 - y_0|}{\varepsilon} + 1} \left(\frac{1 - \delta_{\vec{x}\vec{y}}}{|\vec{x} - \vec{y}|^p} + \delta_{\vec{x}\vec{y}} \right) \leq e K F_{xy}^{(1)}, \sup_{x \in \Lambda} \sum_{y \in \Lambda} |A_{xy}^{(s)}| \leq e \mathcal{O}(1) K,$$

where we used the notation

$$K \equiv \max\{A_1, A_2, \dots, A_6\}, \quad (50)$$

and, for $w \in \mathbb{R}$, $w > 0$,

$$F_{xy}^{(w)} \equiv e^{-w \frac{|x_0 - y_0|}{\varepsilon}} \left[\frac{(1 - \delta_{\vec{x}\vec{y}})}{|\vec{x} - \vec{y}|^p} + \delta_{\vec{x}\vec{y}} \right].$$

Then, fixing $\tau \in T_n$ and the sequence $\{s_{ij}\}$, we get

$$\begin{aligned}
& \sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} \prod_{\{i,j\} \in \tau} |A_{x_i x_j}^{(s_{ij})}| = \sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} \prod_{\{i,j\} \in \tau \setminus \bar{\tau}} |A_{x_i x_j}^{(s_{ij})}| \prod_{\{i,j\} \in \bar{\tau}} |A_{x_i x_j}^{(s_{ij})}| \\
& \leq [e\mathcal{O}(1)K]^{(n-k-2)} \sum_{\substack{x_{i_1}, \dots, x_{i_k} \in \Lambda: \\ x_{i_r} \neq x_{i_q} \forall r, q=1, \dots, k}} \prod_{\{i,j\} \in \bar{\tau}} |A_{x_i x_j}^{(s_{ij})}| \\
& \leq [e\mathcal{O}(1)K]^{(n-k-2)} \sum_{\substack{x_{i_1}, \dots, x_{i_k} \in \Lambda: \\ x_{i_r} \neq x_{i_q} \forall r, q=1, \dots, k}} eKF_{x_1 x_{i_1}}^{(1)} eKF_{x_{i_1} x_{i_2}}^{(1)} \dots eKF_{x_{i_{k-1}} x_{i_k}}^{(1)} eKF_{x_{i_k} x_2}^{(1)}.
\end{aligned}$$

Applying iteratively the inequality (for $w_1 < w_2$)

$$\sum_{\substack{x_i \in \Lambda: \\ x_i \neq x, y}} F_{xx_i}^{(w_1)} F_{x_i y}^{(w_2)} \leq \mathcal{O}(1) F_{xy}^{(w_1)}, \quad (51)$$

which follows from

$$\sum_{\substack{\vec{x}_i \in \mathbb{Z}^d: \\ \vec{x}_i \neq \vec{x}, \vec{y}}} \frac{1}{|\vec{x} - \vec{x}_i|^p} \frac{1}{|\vec{x}_i - \vec{y}|^p} \leq \frac{\mathcal{O}(1)}{|\vec{x} - \vec{y}|^p}, \quad \text{and} \quad \sum_{\substack{z_0 \in \mathbb{R}: \\ z_0 \neq x_0, y_0}} e^{-w_1 \frac{|x_0 - z_0|}{\varepsilon}} e^{-w_2 \frac{|z_0 - y_0|}{\varepsilon}} \leq \mathcal{O}(1) e^{-w_1 \frac{|x_0 - y_0|}{\varepsilon}},$$

(the formula is valid for any $w_1 < w_2$, in specific for $w_1 = 2/3$ and $w_2 = 1$, which we will take here), we get

$$\sum_{\substack{x_1, \dots, x_n \in \Lambda: \\ x_1 = z, x_2 = z', x_i \neq x_j}} \prod_{\{i,j\} \in \tau} |A_{x_i x_j}^{(s_{ij})}| \leq [e\mathcal{O}(1)K]^{(n-1)} F_{zz'}^{(2/3)}. \quad (52)$$

Recall that

$$\sum_{\tau \in T_n} 1 = \sum_{\substack{d_1 + \dots + d_n = 2n-2 \\ d_i \geq 1}} \sum_{\tau \in T_n: \tau \approx (d_1, \dots, d_n)} 1, \quad (53)$$

where the notation $\tau \approx (d_1, \dots, d_n)$ means that the last sum above runs over the trees $\tau \in T_n$ that have fixed incidence indices (d_1, \dots, d_n) . From the Cayley formula

$$\sum_{\substack{\tau \in T_n: \\ \tau \approx (d_1, \dots, d_n)}} 1 = \frac{(n-2)!}{\prod_{i=1}^n (d_i - 1)!}, \quad (54)$$

and, fixing $\tau \in T_n$, we have

$$\sum_{\{i,j\} \in \tau} \sum_{s_{ij}=1}^6 1 = 6^{n-1}.$$

Hence, using (49), we get

$$\begin{aligned}
& \sum_{\substack{R \subset \Lambda: |R| \geq 2 \\ z, z' \in R}} |\rho(R)| e^{|R|} \\
& \leq \sum_{n \geq \mathcal{N}'} \frac{e^n}{(n-2)!} \left(\frac{\varepsilon^{1/2} \gamma^{-1/2} e^{K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6}{3K_3^{\frac{1}{6}} K_4^{\frac{1}{2}}} \right)^n \tilde{K}_3^{-2n+2} \tilde{K}_4^{-2n+2} \\
& \quad \times \sum_{\tau \in T_n} \sum_{\{i,j\} \in \tau} \sum_{s_{ij}=1}^6 \prod_{k=1}^n \Gamma(d_k) [e\mathcal{O}(1)K]^{(n-1)} F_{zz'}^{(2/3)} \\
& \leq \sum_{n \geq \mathcal{N}'} \frac{e^n}{(n-2)!} \left(\frac{\varepsilon^{1/2} \gamma^{-1/2} e^{K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6}{3K_3^{\frac{1}{6}} K_4^{\frac{1}{2}}} \right)^n \tilde{K}_3^{-2n+2} \tilde{K}_4^{-2n+2} \\
& \quad \times [e\mathcal{O}(1)K]^{(n-1)} F_{zz'}^{(2/3)} 6^{n-1} \sum_{\substack{d_1 + \dots + d_n = 2n-2 \\ d_i \geq 1}} \prod_{k=1}^n (d_k - 1)! \frac{(n-2)!}{\prod_{i=1}^n (d_i - 1)!} \\
& \leq \sum_{n \geq \mathcal{N}'} \left(\frac{\varepsilon^{1/2} \gamma^{-1/2} e^{1+K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6}{3K_3^{\frac{1}{6}} K_4^{\frac{1}{2}}} \right)^n \tilde{K}_3^{-2n+2} \tilde{K}_4^{-2n+2} \\
& \quad \times [e\mathcal{O}(1)K]^{(n-1)} F_{zz'}^{(2/3)} 6^{n-1} 4^n,
\end{aligned}$$

where we used the inequality

$$\sum_{\substack{d_1 + \dots + d_n = 2n-2 \\ d_i \geq 1}} 1 = \sum_{\substack{y_1 + \dots + y_n = n-2 \\ y_i \geq 0}} 1 = \binom{2n-3}{n-2} \leq 2^{2n-3} \leq 4^n, \quad (55)$$

with $y_i = d_i - 1$. Hence,

$$\sum_{\substack{R \subset \Lambda: |R| \geq 2 \\ z, z' \in R}} |\rho(R)| e^{|R|} \leq \frac{4\varepsilon^{1/2} \gamma^{-1/2} e^{1+K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6}{3K_3^{\frac{1}{6}} K_4^{\frac{1}{2}}} F_{zz'}^{(2/3)} \quad (56)$$

$$\begin{aligned}
& \times \sum_{n \geq \mathcal{N}^*} \left\{ \frac{8\varepsilon^{1/2} \gamma^{-1/2} e^{2+K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6 \mathcal{O}(1)}{K_3^{\frac{1}{6}} K_4^{\frac{1}{2}} \tilde{K}_3^2 \tilde{K}_4^2} K \right\}^n \\
& = c \left\{ \frac{8\varepsilon^{1/2} \gamma^{-1/2} e^{2+K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6 \mathcal{O}(1)}{K_3^{\frac{1}{6}} K_4^{\frac{1}{2}} \tilde{K}_3^2 \tilde{K}_4^2} K \right\}^{\mathcal{N}^*} F_{zz'}^{(2/3)} \\
& \quad \times \sum_{n=0}^{\infty} \left\{ \frac{8\varepsilon^{1/2} \gamma^{-1/2} e^{2+K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6 \mathcal{O}(1)}{K_3^{\frac{1}{6}} K_4^{\frac{1}{2}} \tilde{K}_3^2 \tilde{K}_4^2} K \right\}^n, \quad (57)
\end{aligned}$$

where $\mathcal{N}^* = \mathcal{N}' - 1$ and

$$c = \frac{4\varepsilon^{1/2} \gamma^{-1/2} e^{1+K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6}{3K_3^{\frac{1}{6}} K_4^{\frac{1}{2}}}.$$

In short, we have proved the following result.

Lemma 6. *If $K = \max\{A_1, A_2, \dots, A_6\}$ is sufficiently small, then,*

$$\varepsilon(K) = \frac{8\varepsilon^{1/2} \gamma^{-1/2} e^{2+K_5 - K_2} K_1^{1/6} (M + 2\zeta c_1)^{1/2} K_6 \mathcal{O}(1)}{K_3^{\frac{1}{6}} K_4^{\frac{1}{2}} \tilde{K}_3^2 \tilde{K}_4^2} K$$

is a positive function and, for any $z \in \Lambda, z' \in \Lambda$ with $z \neq z'$

$$\sum_{\substack{R \subset \Lambda: |R| \geq 2 \\ z, z' \in R}} |\rho(R)| e^{|R|} \leq c[\varepsilon(K)]^{\mathcal{N}^*} F_{zz'}^{(2/3)} = c[\varepsilon(K)]^{\max\{\frac{|z_0 - z'_0|}{\varepsilon}, 1\}} F_{zz'}^{(2/3)}. \quad (58)$$

From the lemma 6, we obtain

Corollary 1.

$$\sup_{x \in \mathbb{Z}^{d+1}} \sum_{R: x \in R} |\rho(R)| e^{|R|} \leq c\mathcal{O}(1)\varepsilon(K).$$

Proof. In fact, as $\rho(R) = 0$ if $|R| = 1$, we have

$$\begin{aligned} \sup_{x \in \mathbb{Z}^{d+1}} \sum_{R: x \in R} |\rho(R)| e^{|R|} &= \sup_{x \in \mathbb{Z}^{d+1}} \sum_{\substack{R: x \in R \\ |R| \geq 2}} |\rho(R)| e^{|R|} \leq \sup_{x \in \mathbb{Z}^{d+1}} \sum_{\substack{z \in \mathbb{Z}^{d+1}: \\ z \neq x}} \sum_{\substack{R: |R| \geq 2 \\ x, z \in R}} |\rho(R)| e^{|R|} \\ &\leq \sup_{x \in \mathbb{Z}^{d+1}} \sum_{z \in \mathbb{Z}^{d+1}: z \neq x} c[\varepsilon(K)]^{\max\{\frac{|x_0 - z_0|}{\varepsilon}, 1\}} F_{xz}^{(2/3)} \leq c\mathcal{O}(1)\varepsilon(K), \end{aligned}$$

since, for $w > 0$,

$$\sum_{z \in \mathbb{Z}^{d+1}: z \neq x} e^{-w \frac{|x_0 - z_0|}{\varepsilon}} \left[\frac{(1 - \delta_{\vec{x}, \vec{z}})}{|\vec{x} - \vec{z}|^p} + \delta_{\vec{x}, \vec{z}} \right] = \mathcal{O}(1),$$

and so $\sum_{z \in \mathbb{Z}^{d+1}: z \neq x} F_{xz}^{(2/3)} \leq \mathcal{O}(1)$. \square

The results above establish the convergence of the cluster expansion for $\varepsilon(K)$ small enough such that $c\mathcal{O}(1)\varepsilon(K) < 1$, i.e., for ζ , M and, mainly, λ large. See lemma 3 and Eqs.(24, 25).

V. DECAY OF TWO-POINT CORRELATION

As it is well known, the convergence of the cluster expansion assures the decay of the correlation functions and lead to direct estimates. We present the main technical details related to the behavior of the truncated two-point function below.

Turning to the expression

$$\tilde{\rho}(R_i) = \prod_{x \in R_i} \int d\nu(\phi_x) (q_{x_1}^{\beta_i^1} + \beta_i^1 l_q) (p_{x_2}^{\beta_i^2} + \beta_i^2 l_p) \sum_{g \in G_R} \prod_{\{x, y\} \in g} (e^{G_{xy}(\phi_x, \phi_y)} - 1), \quad (59)$$

which defines $\tilde{\rho}(R_i)$, with $\beta_i^j = 0$ if $i \neq i_j$, or 1 if $i = i_j$, $l_q = \int q d\nu(\psi)$ and $l_p = \int p d\nu(\psi)$, we note that the index i of the term β_i^j is the same of the polymer R_i , and so $i \in \{1, 2, 3, \dots, n\}$ as $j \in \{1, 2\}$. Consider the expression (34) for $S_2(x_1; x_2)$, and recall that $i_1, i_2 \in \{1, 2, 3, \dots, n\}$. Then, we have two distinct cases: $i_1 = i_2$ or $i_1 \neq i_2$. If $i_1 = i_2$, then $\{x_1, x_2\} \subset R_{i_1}$, $\{x_1, x_2\} \cap R_i = \emptyset \forall i \neq i_1$ and $\beta_{i_1}^1 = \beta_{i_1}^1 = \beta_{i_2}^2 = \beta_{i_2}^2 = 1$. If $i_1 \neq i_2$, then $x_1 \in R_{i_1}$, $x_1 \notin R_{i_2}$, $x_2 \in R_{i_2}$, $x_2 \notin R_{i_1}$, $\{x_1, x_2\} \cap R_i = \emptyset \forall i \notin \{i_1, i_2\}$, $\beta_{i_1}^1 = \beta_{i_2}^2 = 1$, and $\beta_{i_1}^2 = \beta_{i_2}^1 = 0$.

Hence, as $1 = (1 - \delta_{i_1, i_2}) + \delta_{i_1, i_2}$ we rewrite

$$S_2(x_1; x_2) = D_1(x_1, x_2) + D_2(x_1, x_2), \quad (60)$$

where

$$\begin{aligned} D_1(x_1, x_2) &\equiv \sum_{n \geq 1} \frac{1}{n!} \sum_{i_1, i_2=1}^n (1 - \delta_{i_1, i_2}) \sum_{\substack{R_1, \dots, R_n \subset \Lambda, |R_j| \geq 2 \\ R_{i_1} \ni x_1, R_{i_2} \ni x_2}} \phi^T(R_1, \dots, R_n) \tilde{\rho}(R_1) \dots \tilde{\rho}(R_n) \\ &= \sum_{n \geq 2} \frac{1}{(n-2)!} \sum_{\substack{R_1, \dots, R_n \subset \Lambda, |R_j| \geq 2 \\ R_1 \ni x_1, R_2 \ni x_2}} \phi^T(R_1, \dots, R_n) \tilde{\rho}(R_1) \dots \tilde{\rho}(R_n), \end{aligned} \quad (61)$$

$$\begin{aligned} D_2(x_1, x_2) &\equiv \sum_{n \geq 1} \frac{1}{n!} \sum_{i_1, i_2=1}^n \delta_{i_1, i_2} \sum_{\substack{R_1, \dots, R_n \subset \Lambda, |R_j| \geq 2 \\ R_{i_1} \ni x_1, R_{i_2} \ni x_2}} \phi^T(R_1, \dots, R_n) \tilde{\rho}(R_1) \dots \tilde{\rho}(R_n) \\ &= \sum_{n \geq 1} \frac{1}{(n-1)!} \sum_{\substack{R_1, \dots, R_n \subset \Lambda, |R_j| \geq 2 \\ R_1 \supset \{x_1, x_2\}}} \phi^T(R_1, \dots, R_n) \tilde{\rho}(R_1) \dots \tilde{\rho}(R_n), \end{aligned} \quad (62)$$

since, in $D_1(x_1, x_2)$ when $n = 1$ we have $\sum_{i_1, i_2=1}^1 (1 - \delta_{i_1, i_2}) = 0$ and, for any $n > 2$ the sum $\sum_{i_1, i_2=1}^n (1 - \delta_{i_1, i_2})$ leads to $n(n-1)$ equal terms. And, in $D_2(x_1, x_2)$ the sum $\sum_{i_1, i_2=1}^n \delta_{i_1, i_2}$ gives n equal terms.

Thus,

$$|S_2(x_1; x_2)| \leq |D_1(x_1, x_2)| + |D_2(x_1, x_2)|.$$

Comparing (59) with (26), we note that if $R_i \cap \{x_1, x_2\} = \emptyset$ then $\tilde{\rho}(R_i) = \rho(R_i)$. If $R_i \cap \{x_1, x_2\} \neq \emptyset$, we can obtain the result (58) of the lemma 6 for $\tilde{\rho}(R_i)$ by changing $n_k(s)$ and $m_k(s)$ by $n_k(s) + 1$ and $m_k(s) + 1$. With such result, we change $\Gamma(d_k)$ by $\Gamma(d_k + 1)$ in (48) and obtain an extra $\prod_{k=1}^n d_k$, which is bounded by $e^{2(n-1)}$. We use the lemma 5 with $\alpha = \beta = 1$ to bound the factors l_q and l_p in (35). Hence, we can apply the lemma 6 and corollary 1 to estimate $\tilde{\rho}(R_i)$ (changing some multiplicative constants).

Now, let us find an upper bound for the term $|D_1(x_1, x_2)|$. We have

$$|D_1(x_1, x_2)| \leq \sum_{n \geq 2} \frac{1}{(n-2)!} B_n(x_1, x_2), \quad (63)$$

where

$$B_n(x_1, x_2) = \sum_{\substack{R_1, \dots, R_n \subset \Lambda \\ |R_i| \geq 2, x_1 \in R_1, x_2 \in R_2}} |\phi^T(R_1, R_2, \dots, R_n)| \tilde{\rho}(R_1) |\tilde{\rho}(R_2)| \tilde{\rho}(R_3) \dots \tilde{\rho}(R_n)|.$$

Note that in (28), for $n \geq 2$, $\phi^T(R_1, \dots, R_n) > 0$ only if $g(R_1, \dots, R_n) \in G_n$. Thus,

$$\sum_{\substack{R_1, \dots, R_n \subset \Lambda \\ |R_i| \geq 2, x_1 \in R_1, x_2 \in R_2}} |\phi^T(R_1, R_2, \dots, R_n)| [\cdot] = \sum_{g \in G_n} \left| \sum_{\substack{f \in G_n \\ f \subset g}} (-1)^{|f|} \right| \sum_{\substack{R_1, \dots, R_n \subset \Lambda: |R_i| \geq 2 \\ g(R_1, \dots, R_n) = g, x_1 \in R_1, x_2 \in R_2}} [\cdot].$$

By the Rota formula [35], we have

$$\left| \sum_{\substack{f \in G_n \\ f \subset g}} (-1)^{|f|} \right| \leq \sum_{\tau \in T_n: \tau \subset g} 1 \equiv N(g). \quad (64)$$

A proof of the Rota formula above can be found e.g. in Refs. [35] and [28].

We recall now that

$$\sum_{g \in G_n} [\cdot] = \sum_{\tau \in T_n} \sum_{g: \tau \subset g} \frac{1}{N(g)} [\cdot],$$

since in the double sum $\sum_{\tau} \sum_{g \supset \tau}$ each g will be repeated exactly $N(g)$ times.

Thus,

$$B_n(x_1, x_2) \leq \sum_{\tau \in T_n} w_n(\tau, x_1, x_2),$$

where

$$w_n(\tau, x_1, x_2) \equiv \sum_{\substack{R_1, \dots, R_n \subset \Lambda: |R_i| \geq 2 \\ g(R_1, R_2, \dots, R_n) \supset \tau, x_1 \in R_1, x_2 \in R_2}} |\tilde{\rho}(R_1)| |\tilde{\rho}(R_2)| \tilde{\rho}(R_3) \dots \tilde{\rho}(R_n)|.$$

Using now the obvious bound

$$\sum_{R: R \cap R' \neq \emptyset} |\cdot| \leq |R'| \sup_{x \in R'} \sum_{R: x \in R} |\cdot|,$$

and denoting again as $\bar{\tau}$ the subtree of τ which is the unique path joining vertex 1 to vertex 2, and denoting as $I_\tau = \{1, i_1, \dots, i_k, 2\}$ the ordered set of the vertices of $\bar{\tau}$, one can easily check that

$$\begin{aligned}
w_n(\tau, x_1, x_2) &\leq \prod_{i \notin I_\tau}^n \left[\sup_{x \in \mathbb{Z}} \sum_{R_i: x \in R_i} |R_i|^{d_i-1} |\tilde{\rho}(R_i)| \right] \\
&\times \sum_{\substack{R_1, R_{i_1}, \dots, R_{i_k}, R_2: x_1 \in R_1, x_2 \in R_2 \\ R_1 \cap R_{i_1} \neq \emptyset, \dots, R_{i_k} \cap R_2 \neq \emptyset}} |R_1|^{d_1-1} |\tilde{\rho}(R_1)| |R_2|^{d_2-1} |\tilde{\rho}(R_2)| \prod_{\substack{i \in I_\tau \\ i \neq 1, 2}}^n |R_i|^{d_i-2} |\tilde{\rho}(R_i)| \\
&\leq \prod_{i \notin I_\tau}^n \left[\sup_{x \in \Lambda} \sum_{R_i: x \in R_i} (d_i - 1)! |\tilde{\rho}(R_i)| e^{|R_i|} \right] (d_1 - 1)! (d_2 - 1)! \\
&\times \sum_{\substack{R_1, R_{i_1}, \dots, R_{i_k}, R_2: x_1 \in R_1, x_2 \in R_2 \\ R_1 \cap R_{i_1} \neq \emptyset, \dots, R_{i_k} \cap R_2 \neq \emptyset}} |\tilde{\rho}(R_1)| e^{|R_1|} |\tilde{\rho}(R_2)| e^{|R_2|} \prod_{\substack{i \in I_\tau \\ i \neq 1, 2}} (d_i - 2)! |\tilde{\rho}(R_i)| e^{|R_i|},
\end{aligned}$$

since $|R|^n \leq n! e^{|R|}$. Now, note that

$$\sum_{\substack{R_1, R_{i_1}, \dots, R_{i_k}, R_2: x_1 \in R_1, x_2 \in R_2 \\ R_1 \cap R_{i_1} \neq \emptyset, \dots, R_{i_k} \cap R_2 \neq \emptyset}} \leq \sum_{x_{i_0} \in \Lambda} \sum_{x_{i_1} \in \Lambda} \cdots \sum_{x_{i_k} \in \Lambda} \sum_{\substack{R_1 \\ x_1, x_{i_0} \in R_1}} \sum_{\substack{R_{i_1} \\ x_{i_0}, x_{i_1} \in R_{i_1}}} \cdots \sum_{\substack{R_{i_k} \\ x_{i_{k-1}}, x_{i_k} \in R_{i_k}}} \sum_{\substack{R_2 \\ x_{i_k}, x_2 \in R_2}}.$$

Hence, recalling Eq.(58) and applying iteratively the inequality (51) with $w_1 = 1/2$ and $w_2 = 2/3$,

$$\begin{aligned}
&\sum_{\substack{R_1, R_{i_1}, \dots, R_{i_k}, R_2: x_1 \in R_1, x_2 \in R_2 \\ R_1 \cap R_{i_1} \neq \emptyset, \dots, R_{i_k} \cap R_2 \neq \emptyset}} |\tilde{\rho}(R_1)| e^{|R_1|} |\tilde{\rho}(R_2)| e^{|R_2|} \prod_{i \in I_\tau} |\tilde{\rho}(R_i)| e^{|R_i|} \\
&\leq \sum_{x_{i_0} \in \Lambda} \sum_{x_{i_1} \in \Lambda} \cdots \sum_{x_{i_k} \in \Lambda} c[\varepsilon(K)]^{\max\{\frac{|(x_1)_0 - (x_{i_0})_0|}{\varepsilon}, 1\}} F_{x_1 x_{i_0}}^{(2/3)} \cdots c[\varepsilon(K)]^{\max\{\frac{|(x_{i_k})_0 - (x_2)_0|}{\varepsilon}, 1\}} F_{x_{i_k} x_2}^{(2/3)} \\
&\leq [\mathcal{O}(1)]^{k+1} c^{k+2} [\varepsilon(K)]^{\max\{\frac{|(x_1)_0 - (x_2)_0|}{\varepsilon}, k+2\}} F_{x_1 x_2}^{(1/2)},
\end{aligned}$$

since $\varepsilon(K) < 1$ and $|(x_1)_0 - (x_2)_0| \leq |(x_1)_0 - (x_{i_0})_0| + |(x_{i_0})_0 - (x_{i_1})_0| + \cdots + |(x_{i_k})_0 - (x_2)_0|$.

Thus, using the corollary 1 and noting that $|\{1, \dots, n\} \setminus I_\tau| = n - k - 2$,

$$\begin{aligned}
w_n(\tau, x_1, x_2) &\leq (d_1 - 1)! (d_2 - 1)! \left[\prod_{i \notin I_\tau}^n \sup_{x \in \mathbb{Z}} \sum_{R_i: x \in R_i} (d_i - 1)! |\rho(R_i)| e^{|R_i|} \right] \\
&\times \left[\prod_{\substack{i \in I_\tau \\ i \neq 1, 2}}^n (d_i - 2)! \right] c^{k+2} [\varepsilon(K)]^{\max\{\frac{|(x_1)_0 - (x_2)_0|}{\varepsilon}, k+2\}} [\mathcal{O}(1)]^{k+1} F_{x_1 x_2}^{(1/2)} \\
&\leq [\mathcal{O}(1)]^n c^n [\varepsilon(K)]^{\max\{\frac{|(x_1)_0 - (x_2)_0|}{\varepsilon}, n\}} F_{x_1 x_2}^{(1/2)} \prod_{i=1}^n (d_i - 1)!.
\end{aligned}$$

Finally, carrying out the sum over τ (and using, once again, the Cayley formula) we obtain

$$B_n(x_1, x_2) \leq (n - 2)! [4\mathcal{O}(1)]^n [\varepsilon(K)]^{\max\{\frac{|(x_1)_0 - (x_2)_0|}{\varepsilon}, n\}} F_{x_1 x_2}^{(1/2)}.$$

Taking K small enough to make $4\mathcal{O}(1)\varepsilon(K) < 1$, for the contribution of D_1 to the correlations, we get the following bound:

$$|D_1(x_1, x_2)| \leq \sum_{n \geq 2} [4\mathcal{O}(1)]^n [\varepsilon(K)]^{\max\{\frac{|(x_1)_0 - (x_2)_0|}{\varepsilon}, n\}} F_{x_1 x_2}^{(1/2)} \leq \mathcal{O}(1) [\varepsilon(K)]^{\frac{|(x_1)_0 - (x_2)_0|}{\varepsilon}} F_{x_1 x_2}^{(1/2)}. \quad (65)$$

In a similar and much easier way one can also prove a completely analogous bound for $|D_2(x_1, x_2)|$

$$|D_2(x_1, x_2)| \leq \mathcal{O}(1) [\varepsilon(K)]^{\frac{|(x_1)_0 - (x_2)_0|}{\varepsilon}} F_{x_1 x_2}^{(1/2)}. \quad (66)$$

Hence,

$$\begin{aligned}
|S_2(x; y)| &\leq \mathcal{O}(1)[\varepsilon(K)]^{\frac{|x_0-y_0|}{\varepsilon}} F_{xy}^{(1/2)} \leq \mathcal{O}(1)[\varepsilon(K)]^{\frac{|x_0-y_0|}{\varepsilon}} e^{-\frac{|x_0-y_0|}{2\varepsilon}} \left(\frac{1 - \delta_{\vec{x}, \vec{y}}}{|\vec{x} - \vec{y}|^p} + \delta_{\vec{x}, \vec{y}} \right) \\
&\leq \mathcal{O}(1)e^{-m'(K)|x_0-y_0|} \left(\frac{1 - \delta_{\vec{x}, \vec{y}}}{|\vec{x} - \vec{y}|^p} + \delta_{\vec{x}, \vec{y}} \right),
\end{aligned} \tag{67}$$

where, since $\varepsilon(K) < 1$, we write above

$$m'(K) \equiv \frac{-\log[\varepsilon(K)] + 1/2}{\varepsilon} > 0.$$

It is important to remark that the existence of a convergent polymer expansion, such as that presented above, allows us to obtain also a lower bound for the correlations. Roughly, if we write the polymer series as a main term plus corrections, we get the upper bound; and the lower bound is given by the main term minus corrections, see Ref.[30].

In short, the results of this section may be summarized as follows.

Theorem 2. *The two-point function $S_2(x; y)$ (19) of the anharmonic chain of oscillators with discrete times, written as a polymer expansion (34), converges absolutely and uniformly in the volume $|\Lambda|$ (number of sites N and time \mathfrak{T}), for ζ, M, λ large enough. Moreover, $S_2(x; y)$ has the upper bound*

$$|S_2(x; y)| \leq C' e^{-m'|x_0-y_0|} \left(\frac{1 - \delta_{\vec{x}, \vec{y}}}{|\vec{x} - \vec{y}|^p} + \delta_{\vec{x}, \vec{y}} \right).$$

And a similar lower bound follows, with other properly chosen parameters C'' and m'' .

Some short notes are appropriate here.

As described above, the decay in space of the two-point function S_2 is polynomial, and follows the decay of the interparticle interaction $J_{j,\ell}$. As the two-point function is directly related to the heat flow and to the thermal conductivity in these systems given by chains of oscillators, such a result is of direct interest: see e.g. Ref.[16] in which we assumed the space decay of terms in the expression of the heat flow related to the space decay of the interparticle interaction - result which is proved by theorem 2.

An investigation about the precise rate for the exponential decay in time of S_2 (something between m' and m'') may be possible by using standard techniques of constructive field theory related to spectral analysis [29], but it is beyond the aim of the present work. See e.g. Refs. [36, 37] for examples of detailed study of the two and four-point correlations decay in time, via such an approach, in the stochastic Ginzburg-Landau model (a simpler system with nonconservative dynamics and, in these specific works, relaxing to equilibrium).

Finally, we have a remark about the time discretization. Here, as previously described, we work on a lattice with time step ε . To follow the dependence on ε , note that such factors are hidden in some terms, for example, in $|x_0 - y_0|$ (which is a multiple of ε). Hence, in $m'|x_0 - y_0|$, which appears in the theorem above, there is a factor ε in the denominator of the expression for m' , as well as another one in the numerator within $|x_0 - y_0|$. However, we need to say that, considering the whole problem (all expressions and manipulations), if we naively try to recover the original nonlinear model with continuous time by simply taking the limit of ε going to zero, divergences and problems will appear. In short, recovering the continuous limit is not a trivial work. To illustrate such adversity, we note that the expression for the Gaussian measure related to the harmonic part (see Eq.(18)) is well defined and controllable in the continuous limit, but we do not have the inverse of the covariance \mathcal{C} , i.e., the diffusion matrix \mathcal{C} is not invertible in the continuous time limit (and, we recall, \mathcal{C}^{-1} , or a related expression, is well defined and important in the formalism with discrete time). Such trouble is not specific for the present investigation: it is very well known in the study of stochastic processes, see e.g. Ref.[38] and references there in. Moreover, such difficulty in taking the continuous limit is also common in other related problems in physics, as already said: recall, for example, the ultraviolet (UV) limit in Quantum Field Theory. Anyway, in a lattice with a fixed step, we can still obtain a precise description for the heat flow investigation, as we confirm in the next section.

VI. A CONCRETE EXAMPLE: ANALYSIS OF THE ANHARMONIC CHAIN WITH QUARTIC ON-SITE POTENTIAL

Now we turn to the analysis of a concrete and recurrent problem: the heat flow in the anharmonic chain of oscillators with quartic on-site potential. Here, besides the specific quartic on-site potential, we take a model with weak and nearest-neighbor interparticle interactions, and with the inner reservoirs in the self-consistent condition. Precisely,

for the interparticle interaction we take $\mathcal{J}_{ij} \neq 0 \iff i = (j + N) \pm 1$, $|\mathcal{J}_{ij}| \ll 1$; and, for the anharmonic on-site potential, $\mathcal{P}(\phi_j) = \phi_j^4/4$. Moreover, for simplicity, besides the regimes already considered ($m = 1$, $M = 2 + \varepsilon$), we still assume the regime of high anharmonicity and temperature.

The present section is directed toward a twofold aim: first, to show the usefulness of the polymer expansion convergence by describing an interesting result obtained within a perturbative analysis; second, to show the trustworthiness of the discrete time approximation by presenting results which, in comparison with well known numerical computations, are precise.

We need to remark that quite similar results have been already described, by some of the authors, in a previous work [17]. However, at that time, the analysis was carried out in an uncontrolled perturbative approach: the convergence of a related polymer expansion was unknown. For completeness, we repeat some details here.

The heat current, as previously described, follows from Eq.(6)

$$\mathcal{F}_{j,j+1} = \frac{J_{j,j+1}}{2} \langle (\varphi_j - \varphi_{j+1})(\varphi_{j+N} + \varphi_{j+1+N}) \rangle.$$

To analyze it in the steady state, we need to study the averages of $\varphi_j(\mathfrak{T})\varphi_{j+N+1}(\mathfrak{T})$, $\varphi_j(\mathfrak{T})\varphi_{j+N}(\mathfrak{T})$, etc., as $\mathfrak{T} \rightarrow \infty$. That is, we need to evaluate some two-point correlation functions. Recall that, from Eq.(17), the correlations were first written as

$$\langle \varphi_i(\mathfrak{T})\varphi_j(\mathfrak{T}) \rangle = \int \phi_i(\mathfrak{T})\phi_j(\mathfrak{T}) \exp[-W(\phi)] d\mu_{\mathcal{C}},$$

where

$$\begin{aligned} W(\phi) = & \int_0^{\mathfrak{T}} \phi_j(s) \mathcal{J}_{ji}^\dagger \gamma_i^{-1} d\phi_i(s) + \lambda \gamma_i^{-1} P'(\phi)_i(t) d\phi_i(s) + \phi_j(s) \mathcal{J}_{ij}^\dagger \gamma_i^{-1} A_{ik}^0 \phi_k(s) ds + \\ & + \lambda \gamma_i^{-1} P'(\phi)_i(s) A_{ik}^0 \phi_k(s) ds + \frac{1}{2} \phi_{j'}(s) \mathcal{J}_{j'i}^\dagger \gamma_i^{-1} \mathcal{J}_{ij} \phi_j(s) ds + \\ & + \frac{1}{2} \lambda^2 \gamma_i^{-1} (P'(\phi)_i)^2(s) ds + \lambda \gamma_i^{-1} P'(\phi)_i(s) \mathcal{J}_{ij} \phi_j(s) ds, \end{aligned}$$

and the Gaussian measure $d\mu_{\mathcal{C}}$ was given by Eq.(18). Due to exceeding difficulties in the investigation within this first formalism, our idea to perform the computation is resumed in the following strategy (as exhaustively emphasized throughout the paper): we introduce the approximation of discrete times, see Eq.(19), and rearrange the integral representation in terms of a new measure, precisely, a suitable single spin distribution (SSD) with nonlinear parts, as presented in section The Polymer Expansion. In short, we rewrite $\exp[-W(\phi)] d\mu_{\mathcal{C}}$ as $\exp[-\tilde{W}(\phi)] d\nu$. In this properly built SSD $d\nu$, instead of considering the fields ϕ_j and ϕ_i always in separate (as in a usual polymer expansion), we join in the same cell the pairs $\phi_j(s)$ and $\phi_i(s + \varepsilon)$ with $i = j + N$ (of course, $\phi_j(\mathfrak{T})$ and $\phi_i(0)$ do not have pairs). Precisely, our SSD is given by the expression

$$d\nu(\phi_j(s), \phi_{i=j+N}(s + \varepsilon)) = \exp \left\{ \varepsilon \left[-\frac{1}{2} \lambda^2 \gamma_j^{-1} \phi_j^6(s) - \frac{1}{2T_i} \phi_i^2(s + \varepsilon) - \gamma_j^{-1} \lambda \phi_j^3(s) \phi_i(s + \varepsilon) + \dots \right] \right\} d\phi_j(s) d\phi_i(s + \varepsilon) / N, \quad (68)$$

where the dots above describe subdominant terms, N is the normalization, and $\phi_i^2(s + \varepsilon)$ was extracted from $(\phi, \mathcal{D}^{-1}\phi)$, which comes from the harmonic potential related to the Gaussian measure which appeared in the previous formalism. And $\tilde{W}(\phi)$ above is given by subdominant terms that we left behind, both from $\exp[-W(\phi)]$ and $d\mu_{\mathcal{C}}$, after writing the expression for the SSD, i.e.

$$\begin{aligned} \tilde{W}(\phi) = & - \sum_{s,i,j,\dots} \varepsilon \left[\phi_j(s) \mathcal{J}_{ji}^\dagger \gamma_i^{-1} \phi_i(s + \varepsilon) + \phi_j(s) \mathcal{J}_{ji}^\dagger \frac{M_{i-N}}{\gamma_i} \phi_{i-N}(s) \right. \\ & \left. + \frac{1}{2\gamma_i} \phi_{j'}(s) \mathcal{J}_{j'i}^\dagger \mathcal{J}_{ij} \phi_j(s) + \frac{\lambda}{\gamma_i} P'(\phi_{i-N}(s)) \mathcal{J}_{ij} \phi_j(s) + \frac{1}{2} \phi_k(s) \tilde{\mathcal{D}}_{k,k'}^{-1}(s, s') \phi_{k'}(s') \right], \quad (69) \end{aligned}$$

where $\tilde{\mathcal{D}}_{k,k'}^{-1}$ is the quadratic part $\mathcal{D}_{k,k'}^{-1}$ without the terms which are already considered in the SSD.

Note that, essentially, ϕ_i^2 and ϕ_j^6 rule the behavior of the SSD above, and the forthcoming computations. Hence, in our final formalism here, the integral representation for the two-point function is given as product of these SSD (with cells of sites $[j, s]$ and $[i = j + N, s + \varepsilon]$) and the exponential of terms involving the weak interaction J , which couples different cells, and the remaining terms from $(\phi, \mathcal{C}^{-1}\phi)$, which are also small: e.g., for the part involving ϕ_j ,

in the regime of large anharmonicity, rescaling the dominant term $\lambda^2 \phi_j^6$ as $\tilde{\phi}_j^6$ in the s.s.d., this part will involve $\tilde{\phi}_j$ and powers of $1/\lambda$.

Now, we perform a perturbative computation, considering in $\exp\{-\tilde{W}(\phi)\}$ only the terms up to first order, i.e. taking $\exp[-\tilde{W}(\phi)] \approx 1 - \tilde{W}(\phi)$. These leading terms are directly related to the terms which appear in the polymer expansion written as $(e^{G_{x,y}} - 1)$; see section The Polymer Expansion.

Thus, carrying out the computations, we note that a first important contribution is given by

$$\begin{aligned} & \int (\phi_{i-N}(\mathfrak{T})\phi_{i+1}(\mathfrak{T})) \cdot \varepsilon[\lambda\gamma_{i+1}^{-1}\phi_{i+1-N}^3(\mathfrak{T}-\varepsilon)\phi_{i+1}(\mathfrak{T})]_* \\ & \cdot \varepsilon[\lambda\gamma_{i+1}^{-1}\phi_{i+1-N}^3(\mathfrak{T}-\varepsilon)\mathcal{J}_{i+1,i-N}\phi_{i-N}(\mathfrak{T}-\varepsilon)] \\ & \cdot \varepsilon[\phi_{i-N}(\mathfrak{T}-\varepsilon)\mathcal{C}_{i-N,i-N}^{-1}(\mathfrak{T}-\varepsilon,\mathfrak{T})\phi_{i-N}(\mathfrak{T})]d\tilde{\nu}(\phi) \\ & \sim c'(\varepsilon)J\frac{1}{\lambda^{4/3}}\frac{T_{i+1}^{2/3}}{T_i}, \end{aligned}$$

where $[\cdot]_*$ above comes from the “cross” term in the s.s.d.; $d\tilde{\nu}$ is the main part of the s.s.d. (involving ϕ_i^2 and ϕ_j^6); and c' is a numerical factor. And, a second important contribution comes from terms similar to

$$\begin{aligned} & \int (\phi_{i-N}(\mathfrak{T})\phi_{i+1}(\mathfrak{T})) \cdot \varepsilon[\phi_{i-N}(\mathfrak{T}-\varepsilon)\mathcal{J}_{i-N,i+1}^\dagger\gamma_{i+1}^{-1}\phi_{i+1}(\mathfrak{T})] \\ & \cdot \varepsilon[\phi_{i-N}(\mathfrak{T}-\varepsilon)\mathcal{C}_{i-N,i-N}^{-1}(\mathfrak{T}-\varepsilon,\mathfrak{T})\phi_{i-N}(\mathfrak{T})]d\tilde{\nu}(\phi) \\ & \sim c''J\frac{1}{\lambda^{4/3}}\frac{1}{T_i^{1/3}}. \end{aligned}$$

Hence, summing up all leading terms (with $\mathfrak{T} \rightarrow \infty$), and considering a small difference between T_{i+1} and T_i (such that $T_{i+1}^\alpha - T_i^\alpha \approx \alpha T_i^{\alpha-1}(T_{i+1} - T_i)$), we get

$$\mathcal{F}_{j,j+1} \approx -c\frac{J^2}{\lambda^{4/3}}\frac{1}{T_j^{4/3}}(T_{j+1} - T_j). \quad (70)$$

Now, after obtaining the expression for $\mathcal{F}_{j,j+1}$, the computation of the heat current in terms of the temperatures at the boundaries is straightforward. The self-consistent condition in the steady state says that there is no neat heat flows from the inner reservoirs to the system, i.e., it gives

$$\mathcal{F}_{1,2} = \mathcal{F}_{2,3} = \dots = \mathcal{F}_{N-1,N} \equiv \mathcal{F}. \quad (71)$$

These equations together with Eq.(70) give us

$$\begin{aligned} \mathcal{F}(\mathcal{C}T_1^\alpha) &= T_1 - T_2 \\ \mathcal{F}(\mathcal{C}T_2^\alpha) &= T_2 - T_3 \\ &\dots = \dots \\ \mathcal{F}(\mathcal{C}T_{N-1}^\alpha) &= T_{N-1} - T_N. \end{aligned}$$

Summing up the expressions above, we find

$$\mathcal{F} = \mathcal{K}\frac{(T_1 - T_N)}{N - 1},$$

where

$$\mathcal{K} = \{\mathcal{C}T_1^\alpha + \mathcal{C}T_2^\alpha + \dots + \mathcal{C}T_{N-1}^\alpha\}^{-1} \cdot (N - 1),$$

with $\mathcal{C}^{-1} = c(\varepsilon)\frac{J^2}{\lambda^{4/3}}$, $\alpha = 4/3$. For a small gradient of temperature, i.e., if $T_j \approx T$, we have the Fourier's law in the chain with thermal conductivity

$$\mathcal{K} \sim c(\varepsilon)\frac{J^2}{\lambda^{4/3}T^{4/3}}. \quad (72)$$

We emphasize the interest of such result. For the case of a strong anharmonic on-site potential, the effects of the internal reservoirs become less important, and one expects a system with behavior close to that observed in a chain with thermal baths only at the boundaries. Exhaustive computer simulations have been already carried out for these anharmonic chains with quartic potential and thermal baths only at the ends, and they give a thermal conductivity $\mathcal{K} \approx 1/T^{1.35}$ [39, 40], essentially the same result obtained by the perturbative computation within our approach with the approximation of discrete times. Similar results, still for the anharmonic chain with reservoirs only at the boundaries, are presented in Ref.[41], here obtained by using nonequilibrium molecular dynamics. It would be very interesting to make a comparison between our findings and numerical simulations in the original anharmonic model with inner noises, but we do not know any numerical result in the literature for such model, and we have to leave this work for the experts in computer techniques.

VII. FINAL REMARKS

We conclude with some comments and remarks.

First, as example of the trustworthiness of the perturbative analysis within the integral formalism, we recall that, when restricted to the easier case of harmonic chain of oscillators with the self-consistent condition, perturbative computations within our integral representation, with continue time and without simplification in the quadratic term (see Ref.[14]), reproduces the well known result that is described in Ref.[21], there obtained by a completely different method. That is, within our integral approach, we have proved that Fourier's law holds in the chain of harmonic oscillators with self-consistent inner baths, and we obtained the expression for the thermal conductivity as that derived by other methods.

The behavior of the two-point function for a system with interparticle interaction with polynomial decay, what is proved here, allows us to establish the heat flow between different sites in a chain with interparticle interaction beyond nearest-neighbor sites. As already said, that is a problem of physical interest: for example, in a recent work [16], one of the authors and a collaborator, by assuming the heat flow behavior (correlation decay in space), which is proved in the present paper, show that the existence of interparticle interactions beyond nearest neighbors may increase by thousand times the thermal rectification in a graded chain, and may also avoid the decay of such rectification with the system size, which are important properties for the theoretical study and even experimental fabrication of thermal diodes. The effect of long range interactions increasing the thermal rectification of anharmonic crystals has been confirmed, by computer simulations, even in anharmonic crystals without inner self-consistent baths [42].

It is also worth to mention that, in a previous work within a perturbative computation (which could not be rigorously justified at that time, before the present results), we describe nontrivial properties of the heat flow in an inhomogeneous anharmonic chain, namely, thermal rectification and negative differential thermal resistance [18].

To conclude, we believe that, even though within an effective model (anharmonic chain of oscillators with inner stochastic reservoirs) and an approximation in the integral representation (discrete times), the approach and results presented here may be of great utility in the qualitative understanding of the heat flow properties in the steady state of high anharmonic systems submitted to different temperatures. In particular, the existence of a convergent polymer expansion making possible a perturbative investigation is of usefulness in the study of systems with long range interaction, and also in the analysis of inhomogeneous and asymmetric models, in which, the important phenomenon of thermal rectification appears.

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Appendix A: On the used approximations

As an argument, beyond the technical reasons already mentioned, to support the study of the system with time regularization (i.e., without short times), we recall that most of the physical research problems related to similar models involve questions about properties of the steady state, reached as $\mathfrak{T} \rightarrow \infty$. Moreover, from numerical studies of similar dynamical problems carried out by physicists [43], lower frequencies seem to dominate the transport on a large scale (the scale of the whole chain). That is, it seems that the time regularization does not spoil the main features of the original problem, related to heat flow properties in the steady state.

In relation to the other approximation assumed in the present paper, namely, the modified covariance (18), we show below that it is, indeed, the main part to the original harmonic interaction.

The expressions for the covariance are given by Eqs.(13,14). First, we note that the replacement of $C(t, t)$ by $C \equiv C(\infty, \infty)$ does not change the steady heat current for the harmonic case, as shown in Ref.[14]. And so, we study the covariance with such a replacement. Furthermore, we may write $\exp[-(t-s)A_0]$, $t \geq s$, as

$$\exp(-|\tau|A^0) = e^{-|\tau|\alpha} \left[\cosh(\tau\rho) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(|\tau|\rho)}{\rho} \begin{pmatrix} \alpha & I \\ -\mathcal{M} & -\alpha \end{pmatrix} \right], \quad (\text{A1})$$

where $\tau = t - s$, $\alpha = \zeta/2$, $\rho = (\alpha^2 - M)^{1/2}$; and a similar expression given by the transposed matrix follows for negative τ , see Ref.[21]. To proceed, we may introduce the discrete times and study the Fourier transform $\hat{C}_*(p_0)$, where

$$C_*(\tau = t - s) = \begin{cases} e^{-(t-s)A^0} C, & t \geq s, \\ C e^{-(s-t)A^{0\dagger}}, & t \leq s. \end{cases} \quad (\text{A2})$$

Then, with the inverse Fourier transform of $\hat{C}_*^{-1}(p_0)$, we obtain an expression (but huge and unclear) for \mathcal{D}^{-1} . Instead of that, we propose the use of an approximated expression (but with the main part of \mathcal{C}), derived as described below.

To begin, we consider the regime of strong pinning. For strong pinning $M > \alpha^2$, we have $\cosh(\tau\rho) = \cos(\tau\tilde{\rho})$ and $\sinh(\tau\rho)/\rho = \sin(\tau\tilde{\rho})/\tilde{\rho}$, where $\tilde{\rho}$ is given by $\tilde{\rho} = (M - \alpha^2)^{1/2}$. Taking the Fourier transform of $\exp[-|\tau|\alpha] \cos(\tilde{\rho}\tau)$, we have

$$\exp[-|\tau|\alpha] \cos(\tilde{\rho}\tau) = 2\alpha \left\{ \frac{M + p_0}{(M + p_0^2)^2 - 4(M - \alpha^2)p_0^2} \right\} \equiv \hat{D}(p_0),$$

where we used continue times above just for ease of computation; discrete times lead to a similar expression with $1 - \cos(p_0)$ replacing p_0^2 . The second part of $\exp[-|\tau|A^0]$, with $\sin(\tau\tilde{\rho})/\tilde{\rho}$, involves a matrix whose diagonal terms will be very small for large $\tilde{\rho}$. Due to M , the off diagonal terms are not small in principle, but they are related to the “crossed” part qp . These terms will be insignificant later (they will be small) in the interacting model controlled by the cluster expansion: after a scaling, qp will involve a small factor $1/\lambda^{1/3}$ and it will be easily controlled in the case of large λ . Here, for simplicity, we ignore this off diagonal part. Hence, choosing e.g. $M = 3\alpha^2$, we have $\hat{D}^{-1}(p_0) \simeq \frac{M}{2\alpha} + \frac{c}{\alpha} p_0^2$, where c is a numerical factor.

In short, by assuming the approximation above (discarding the off diagonal terms in the original Gaussian covariance), and still taking discrete times in a lattice with spacement ε , we can write the Gaussian measure as proposed in Eq.(18).

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